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Optimal Calibration-Free Observable for the Nucleon-Coupling Ratio in a Dual-Alkali Comagnetometer for Dark Matter Searches

Yossi Rosenzweig[†], Yevgeny Kats[†], Eli Sarid, Menachem Givon, Yonathan Japha, Ron Folman

Department of Physics, Ben-Gurion University of the Negev, Beer-Sheva 8410501, Israel

yosefros@post.bgu.ac.il (Y.R.); katsye@bgu.ac.il (Y.K.); esarid@bgu.ac.il (E.S.); japhay@bgu.ac.il (Y.J.)

[†] These authors contributed equally to this work.

Abstract

A dual-alkali single-cell ^{87}Rb – ^{39}K – ^3He comagnetometer can read an axion-like dark matter signal through two optical rotation channels, encoding the ratio $\mathcal{R} = \tilde{\zeta}_n / \tilde{\zeta}_p$ of the field's neutron and proton spin couplings in their relative response. The inter-species phase difference $\Delta\varphi$ has been proposed as a calibration-free readout that is sensitive to $\mathcal{R}$. Treating the extraction of $\mathcal{R}$ as a statistical estimation problem, we show that the estimate of $\mathcal{R}$ depends on the two channels only through their complex ratio, which splits into $\Delta\varphi$ and an amplitude ratio, of which only $\Delta\varphi$ is insensitive to the relative gain and hence calibration free. For our choice of comagnetometer parameters, above ~ 100 Hz the phase difference alone captures most of the coupling-ratio information. At lower frequencies $\Delta\varphi$ is no longer near-sufficient. Below ~ 40 Hz, the amplitude ratio would improve the precision on $\mathcal{R}$ by a factor of $\gtrsim 2$. Recovering that information, however, requires the relative gain to be known sufficiently accurately, so $\Delta\varphi$ remains the robust observable even where it is not the most informative one.

Keywords: ultralight dark matter; axion-like particles; comagnetometers

1. Introduction

The nature of dark matter remains one of the central open problems in physics: its gravitational presence is firmly established, yet its particle identity and its couplings to Standard Model fields are largely unconstrained [1]. One class of candidates is ultralight pseudoscalar (axion-like) particles [2–4]. Because such light particles need enormous occupation numbers to account for the local dark matter density, they behave as a classical field oscillating coherently over $\sim 10^6$ periods of their Compton frequency [5,6]. Such a field generically couples derivatively to the axial-vector currents of Standard Model fermions [5]; in the non-relativistic limit this acts on fermion spins like a pseudomagnetic field, but with couplings that are generally not proportional to the magnetic moments. Alkali magnetometers and alkali–noble-gas comagnetometers are therefore leading probes of this and similar scenarios [7–15].

If a candidate signal is observed, a decisive follow-up question is how the exotic field couples to the different subatomic particles—protons, neutrons, and electrons—since the pattern of couplings discriminates among microscopic models. Because the electronic structure and the fractional proton and neutron spin content differ among elements and isotopes [16], a field with given microscopic couplings acts on different atomic species through different effective strengths. Earlier work showed how to disentangle these couplings using a *set* of distinct comagnetometers [17]. Ref. [18] proposed instead a single-cell scheme, reviewed below, in which two alkalis interact with the same field. While an amplitude combination of the channels cancels the magnetic background, thus enabling the detection of weaker signals, the phase difference can be used to extract the coupling ratio. The present paper extends ref. [18] to the question of how well a candidate signal can be characterized, retaining the response model, baseline operating point, and noise budget of that work.

The remainder of the paper is organized as follows. Section 2 reviews the dual-alkali scheme of ref. [18] and the response model used throughout. Section 3 formulates the extraction of $\mathcal{R}$ as an estimation problem, with the unknown signal amplitude and phase eliminated from the inference. Section 4 establishes three structural observations that fix how $\mathcal{R}$ is extracted from the two channels: a single channel is blind to $\mathcal{R}$; the coupling-ratio information is carried by the complex inter-channel ratio; and its phase and amplitude carry complementary parts of the information, of which only the phase is calibration free. Section 5 evaluates the attainable precision and the sufficiency of $\Delta\varphi$ as a function of frequency and $\mathcal{R}$. Section 6 derives the tolerances on the operating point and on instrumental systematics and shows how they can be met. Section 7 summarizes the results. Appendix A proves the reduction of the two-channel Fisher information used in the body.

2. The Dual-Alkali Correlated Measurement

Ref. [18] considers a single vapor cell containing two alkali species, ^{87}Rb and ^{39}K , together with ^3He , polarized jointly by hybrid spin-exchange optical pumping [19,20]. The two alkalis are probed simultaneously by two overlapping, linearly polarized beams, each detuned ~ 0.1 THz to the blue of its own D1 line (770.1 nm for ^{39}K , 795.0 nm for ^{87}Rb). After traversing the cell, each probe acquires a Faraday rotation angle proportional to the transverse spin polarization P_x^j of the alkali it addresses [21]. The two beams are separated by a beamsplitter, passed through a narrow-band wavelength filter in each polarimetry arm, and demodulated by a shared lock-in amplifier, so that the channels share a common phase origin. The three species evolve under coupled Bloch equations [17,22,23] in which each species' exotic coupling is

$$\tilde{\zeta}_j = \eta_j (\sigma_n^j \tilde{\zeta}_n + \sigma_p^j \tilde{\zeta}_p), \quad (1)$$

where $\tilde{\zeta}_{n,p}$ are the underlying couplings to neutron and proton spin, and $\sigma_{n,p}^j$ describe the nuclear spin content of the atomic species j provided in Table 1. The prefactor η_j equals $q_j - 1$ for the alkalis, where q_j is the nuclear slowing-down factor, and 1 for the noble gas. We follow ref. [18] by considering an example in which the field couples to nucleon spins but not to electron spin,¹ so the single quantity that characterizes the internal structure of the coupling is the ratio

$$\mathcal{R} \equiv \frac{\tilde{\zeta}_n}{\tilde{\zeta}_p}. \quad (2)$$

Table 2 collects the parameters and noise budget of ref. [18].

The central observation is that a magnetic field acts on both alkalis through the (essentially common) electronic gyromagnetic ratio γ_e , whereas an exotic field acts through the species-dependent couplings of Equation (1). Both fields also reach the alkalis indirectly, through the ^3He , whose driven magnetization couples back to ^{87}Rb and ^{39}K via their unequal Fermi contact factors; this shared route is therefore species dependent for either field. It is strong at low frequency and rolls off as the ^3He can no longer follow the drive. In a standard comagnetometer [27,28], the effects of low-frequency transverse magnetic fields are canceled by operating at the self-compensation point, reached by applying a longitudinal magnetic field at B_c , while the effects of exotic fields survive. With the dual-readout approach of ref. [18], which relies only on the two alkalis sharing the same field, the magnetic field effects can also be subtracted at high frequencies while leaving generic exotic signals unsuppressed.

Table 1. Fractional contributions of the neutron and proton spins σ_n^j and σ_p^j to the total nuclear spin of the three species of the cell, in the convention of ref. [16].

Species	σ_n	σ_p	Refs.
^{39}K	0.034	−0.131	[16,24]
^{87}Rb	0.083	0.251	[25]
^3He	0.87	−0.027	[16,26]

Table 2. Baseline operating parameters and noise budget of ref. [18], adopted throughout this paper. The noise entries are one-sided amplitude spectral densities referred to an equivalent magnetic field.

Cell temperature	T	180 °C
^3He number density	n_{He}	3 amg
Alkali number densities	$n_{\text{Rb}}, n_{\text{K}}$	$4.0, 5.45 \times 10^{13} \text{ cm}^{-3}$
Equilibrium longitudinal polarizations	$p_z^{\text{Rb}}, p_z^{\text{K}}, p_z^{\text{He}}$	0.46, 0.45, 0.047
Alkali relaxation rates	$\Gamma_{\text{Rb}}, \Gamma_{\text{K}}$	600, 600 Hz
Optical pumping rate (^{87}Rb only)	R_p	1200 Hz
Compensation weighting	α	0.578
Correlated magnetic background, $f > 20$ Hz	$\sqrt{S_B}$	$6.2 \text{ fT} / \sqrt{\text{Hz}}$
Differential readout floor at 100 Hz	$\tilde{\sigma}$	$0.22 \text{ fT} / \sqrt{\text{Hz}}$

An observable of interest is the inter-species phase difference of the alkali transverse spin polarizations P_x^j at the angular frequency $\omega = 2\pi f$ of the candidate signal,

$$\Delta\varphi(\omega; \mathcal{R}) \equiv \varphi_{\text{K}}(\omega; \mathcal{R}) - \varphi_{\text{Rb}}(\omega; \mathcal{R}), \quad \varphi_j \equiv \arg P_x^j(\omega; \mathcal{R}). \quad (3)$$

For a magnetic drive, $\Delta\varphi$ is at the milliradian level across the rejection band, which the differing alkali Larmor frequencies confine to tens of Hz and above. Above ~ 50 Hz the exotic response phase difference is one to three orders of magnitude above that baseline. Because the two probe beams are detuned to the same side of their respective lines, the proportionality factor between the measured optical rotation and P_x^j is real and positive for both channels [18], and $\Delta\varphi$ inherits a *calibration-free* character: it is immune to probe intensity, photodetector responsivity, cell length, and correlated density fluctuations.

Two further features of ref. [18] are important to mention. First, the longitudinal compensation field cannot self-compensate both alkalis at once, because their Fermi contact factors differ; it is parameterized as a weighted compromise (Equation (6) of ref. [18]) controlled by a single parameter α , and throughout this paper we adopt the baseline operating point $\alpha = 0.578$ along with the other system parameters of ref. [18]. Second, at a pure-proton coupling ($\mathcal{R} = 0$) the destructive interference among the three exotic drive channels—direct, ^3He -mediated, and alkali–alkali spin exchange—carves an *antiresonance* in the ^{39}K exotic response at ~ 157 Hz and in the ^{87}Rb response at ~ 131 Hz [18]. Both antiresonances persist, at coupling-dependent frequencies, over a range of $\mathcal{R}$ about this value. Near these frequencies the two alkali phases wind in opposite senses, so $\Delta\varphi$ accumulates a single rapid swing that encodes $\mathcal{R}$ (Figure 1). Distinct phase differences also arise where no antiresonance occurs. Mapping $\Delta\varphi$ across the coupling-ratio–frequency plane acts as a matched filter for the coupling ratio. The full map will be presented in Section 5.

3. The Estimation Problem

Ref. [18] establishes $\Delta\varphi$ as a practical, calibration-free readout that carries information about $\mathcal{R}$. It does not, however, address a logically prior question: among *all* observables one could build from the two channels, which one extracts $\mathcal{R}$ with the smallest statistical variance, and how much of that optimal information does the calibration-free $\Delta\varphi$ actually contain? That is the question we answer here.

Each measurement channel $j \in \{^{39}\text{K}, ^{87}\text{Rb}\}$ records a complex demodulated amplitude

$$z_j = \mathcal{N} \mathcal{K}_j T_j^b(\mathcal{R}) + n_j, \quad (4)$$

where $\mathcal{K}_j$ is the channel’s optical-rotation gain (the real, positive prefactor of Equation (1) of ref. [18]) and $T_j^b(\mathcal{R})$ is the exotic transverse-spin transfer function.² The factor $\mathcal{N}$ is the external field, carrying unknown amplitude and phase, which drives both channels. It is therefore a common complex nuisance parameter. The readout noise n_j for each channel is circular: its two quadratures—the real

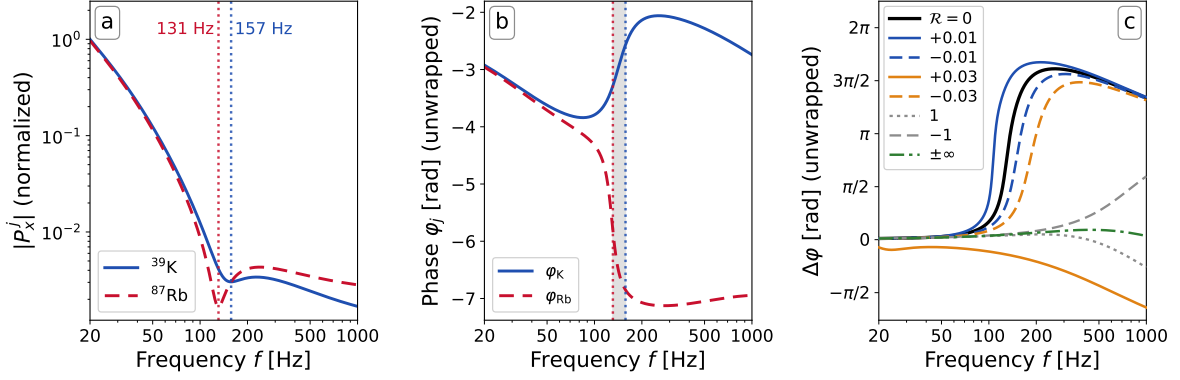


Figure 1. Antiresonances in the exotic response of the two alkalis, and the differential-phase swings they produce. (a) Magnitude of the exotic response $|P_x^j|$ for $j \in \{^{39}\text{K}, ^{87}\text{Rb}\}$ at a pure-proton coupling ($\mathcal{R} = 0$), normalized to its largest value in the band: the destructive interference carves an antiresonance at 157 Hz in ^{39}K and at 131 Hz in ^{87}Rb (dotted vertical lines). (b) The corresponding unwrapped response phases ϕ_j . Across the interval between the two antiresonances (shaded) they wind in opposite senses, ^{39}K advancing by $+0.7$ and ^{87}Rb by -1.0 rad. (c) The resulting differential phase $\Delta\phi$ of Equation (3), unwrapped, for a family of small $|\mathcal{R}|$ of either sign, for the equal mixtures $\mathcal{R} = \pm 1$, and for the pure-neutron limit $\mathcal{R} = \pm\infty$. The swing is a single rapid step that both slides and shrinks with the coupling ratio. The ^{39}K antiresonance moves from 235 Hz at $\mathcal{R} = -0.03$ through 157 Hz at $\mathcal{R} = 0$ to 128 Hz at $\mathcal{R} = +0.01$, and disappears just above $\mathcal{R} = +0.03$, where the step is already gone. The excursion is reduced from 5.0 rad at $\mathcal{R} = 0$ to 1.8, 1.0 and 0.26 rad at $\mathcal{R} = -1, +1$ and $\pm\infty$, where no antiresonance is left either.

and imaginary parts of the demodulated signal—are independent white Gaussian processes of equal one-sided amplitude spectral density σ_j , so that in a bandwidth Δf each has root-mean-square $\sigma_j \sqrt{\Delta f}$.³ The bandwidth Δf is the reciprocal of the coherent integration time, and no smaller than the linewidth of the field, $\sim 10^{-6} f$ for a virialized dark matter field [5].

The response is affine in the coupling ratio, i.e., it has the form $T_j^b(\mathcal{R}) = A_j + \mathcal{R} B_j$, since $\mathcal{R}$ enters only through the per-species drive $\xi_j(\mathcal{R}) \propto \sigma_n^j \mathcal{R} + \sigma_p^j$ of Equation (1). The ratio $\mathcal{R} = \xi_n / \xi_p$ may lie anywhere on the real axis. As it diverges for a pure-neutron coupling ($\xi_p \rightarrow 0$), it is a poor coordinate near that limit. We will therefore use the singularity-free *coupling angle* θ , defined by $(\xi_p, \xi_n) = \xi (\cos \theta, \sin \theta)$, where the overall strength ξ is absorbed into the amplitude $\mathcal{N}$ of Equation (4) and θ carries the coupling ratio alone, so that $\mathcal{R} = \tan \theta$. The range $\theta \in [0, \pi)$ covers every physical coupling ratio exactly once: pure proton at $\theta = 0$, equal couplings at $\theta = \pi/4$, pure neutron at $\theta = \pi/2$, and opposite-sign couplings ($\mathcal{R} < 0$) for $\theta \in (\pi/2, \pi)$. The response is then

$$T_j^b(\theta) = A_j \cos \theta + B_j \sin \theta, \quad (5)$$

where A_j is the pure-proton and B_j the pure-neutron response of channel j .

We will work throughout in terms of the standard estimation theory notions of Fisher information and the Cramér–Rao bound [29,30]. The Fisher information I_λ for a parameter λ is the expected sharpness of the log-likelihood peak at the true parameter value, $I_\lambda = -\mathbb{E}[\partial_\lambda^2 \ln L]$, equivalently the noise-ensemble variance of the first derivative $\partial_\lambda \ln L$: the more sharply peaked the log-likelihood, the more the data constrain the parameter. Its reciprocal bounds the variance of any unbiased estimator—the Cramér–Rao bound, $\sigma_\lambda^2 \geq 1/I_\lambda$. Throughout we quote the resulting standard deviation, $\sigma_\lambda = 1/\sqrt{I_\lambda}$; it is the attainable precision in the small-noise regime, and the Cramér–Rao lower bound otherwise. The Fisher information values below are written with the spectral densities in place of variances, i.e., for a 1 Hz band. A band Δf divides them by Δf , which cancels from every information ratio and every precision quoted at fixed amplitude signal-to-noise ratio (SNR).

4. Structural Observations on the Extraction of $\mathcal{R}$

The following three observations about the dual-readout setup fix how $\mathcal{R}$ is extracted from the two channels and the standing of $\Delta\varphi$ in that extraction.

4.1. A Single Channel Is Blind to $\mathcal{R}$

With one complex measurement and $\mathcal{N}$ free, $\mathcal{R}$ and $\mathcal{N}$ are unidentifiable: any apparent change in $\mathcal{R}$ can be reabsorbed into a redefinition of $\mathcal{N}$. Extracting $\mathcal{R}$ fundamentally requires both channels.

4.2. The Coupling-Ratio Information Is Carried by the Complex Inter-Channel Ratio

Since the common unknown factor $\mathcal{N}$ enters both channels alike, it drops out of the ratio z_K/z_{Rb} , leaving a quantity that is independent of $\mathcal{N}$ but still fully determined by $\mathcal{R}$. This ratio is complex; we work with its logarithm

$$\mathcal{Z} \equiv \ln \left(\frac{T_K^b}{T_{Rb}^b} \right), \quad (6)$$

which turns the multiplicative amplitude/phase structure into additive real and imaginary parts— $\text{Re } \mathcal{Z} = \ln |T_K^b/T_{Rb}^b|$ the log-amplitude ratio and $\text{Im } \mathcal{Z} = \varphi_K - \varphi_{Rb} = \Delta\varphi$ the differential phase—so that the amplitude and phase observables sit on the two Cartesian axes and, as shown below, split the information additively. With independent readout noise in the two channels, each quadrature of the measured $\mathcal{Z}$ has variance $w/|\mathcal{N}|^2$, with

$$w = \left(\frac{\sigma_K}{\mathcal{K}_K |T_K^b|} \right)^2 + \left(\frac{\sigma_{Rb}}{\mathcal{K}_{Rb} |T_{Rb}^b|} \right)^2. \quad (7)$$

The unknown drive $\mathcal{N}$ is removed by profiling: the likelihood is maximized over $\mathcal{N}$ at each value of $\mathcal{R}$. Doing so on the full two-channel likelihood—exact for Gaussian, independent, circular per-channel noise—gives, after an algebraic (Lagrange) reduction,⁴ the Fisher information for $\mathcal{R}$ per unit $|\mathcal{N}|^2$,⁵

$$I_{\mathcal{R}}^{\mathcal{Z}} = \frac{|\partial_{\mathcal{R}} \mathcal{Z}|^2}{w}. \quad (8)$$

Profiling also confines the $\mathcal{R}$ -dependence of the log-likelihood to a single term, which a common rescaling $z_j \rightarrow c z_j$ multiplies by $|c|^2$; the maximum-likelihood estimate of $\mathcal{R}$ is therefore a function of z_K/z_{Rb} alone at any SNR, the amplitude setting only the sharpness of the likelihood. The ratio is not, however, a sufficient statistic away from the small-noise limit, since data with the same ratio but different amplitudes constrain $\mathcal{R}$ differently; it is in this sense that the complex ratio is the optimal observable, and $\Delta\varphi$ the optimal calibration-free one.

The above result presumes the two channels' noise is dominated by *independent*, per-channel readout noise. The dominant *correlated* disturbance, the magnetic background, requires a different treatment. We convert each channel's readout to an equivalent magnetic field by dividing by its own magnetic transfer function T_j^B , thus defining the *field-referred* data $x_j \equiv z_j/(\mathcal{K}_j T_j^B)$. In these units the per-channel readout noise considered above reads $\tilde{\sigma}_j \equiv \sigma_j/(\mathcal{K}_j |T_j^B|)$. Because the magnetic background is common to both channels, it adds a single fully correlated variance S_B on top of the independent per-channel variances $\tilde{\sigma}_j^2$. Their sum $\tilde{\sigma}^2 \equiv \tilde{\sigma}_K^2 + \tilde{\sigma}_{Rb}^2$ is the readout variance of the differential combination $x_K - x_{Rb}$, in which the common background cancels identically; it is the instrument's own noise, in equivalent-field units, and we will use it to normalize S_B . The field-referred data (a two-dimensional vector) are then

$$x = \mathcal{N}s + (\text{noise}), \quad (9)$$

with $s_j = T_j^b/T_j^B$ and the noise described by the covariance matrix

$$C = \begin{pmatrix} \tilde{\sigma}_K^2 + S_B & S_B \\ S_B & \tilde{\sigma}_{Rb}^2 + S_B \end{pmatrix}. \quad (10)$$

The field-referred response inherits the affine form of Equation (5), $s_j(\theta) = a_j \cos \theta + b_j \sin \theta$, with $a_j \equiv A_j/T_j^B$ and $b_j \equiv B_j/T_j^B$ the field-referred pure-proton and pure-neutron responses. Profiling the common drive $\mathcal{N}$ out of the two-channel likelihood gives the Fisher information

$$I_{\mathcal{R}}^x(S_B) = (\partial_{\mathcal{R}} s)^\dagger C^{-1} \partial_{\mathcal{R}} s - \frac{|s^\dagger C^{-1} \partial_{\mathcal{R}} s|^2}{s^\dagger C^{-1} s}. \quad (11)$$

It is the $\mathcal{R}$ -sensitivity of the signal weighted by the inverse noise C^{-1} , minus the part of that sensitivity parallel to the signal itself—a complex rescaling, which a change in $\mathcal{N}$ would mimic and profiling therefore removes. Carrying out the Lagrange reduction (Appendix A) gives

$$I_{\mathcal{R}}^x(S_B) = \frac{|\partial_{\mathcal{R}} \mathcal{Z}|^2}{w_{\text{eff}}}, \quad w_{\text{eff}} = w + S_B \left| \frac{1}{s_K} - \frac{1}{s_{Rb}} \right|^2. \quad (12)$$

Adding the correlated background therefore only rescales the overall information about $\mathcal{R}$.⁶

Panels (a,b) of Figure 2 show the effect of S_B on the Fisher information and the resulting precision. The operating point is taken from the noise values of ref. [18]: above 20 Hz the correlated magnetic background is $6.2 \text{ fT}/\sqrt{\text{Hz}}$, well above the readout floor, whose field-referred value at 100 Hz is $\tilde{\sigma} \approx 0.22 \text{ fT}/\sqrt{\text{Hz}}$, so $S_B/\tilde{\sigma}^2 = (6.2/0.22)^2 \approx 800$ there. That ratio is frequency dependent, since $\tilde{\sigma}_j \propto 1/|T_j^B(f)|$ inherits the magnetic roll-off, a factor of ~ 5 between 30 and 200 Hz, placing the same background at ≈ 5100 at 30 Hz and ≈ 210 at 200 Hz: the vertical lines in the two panels. Panel (c) of that figure maps the resulting penalty $\sqrt{w_{\text{eff}}/w}$, read along this operating locus, across the whole plane of coupling angle and frequency. Over most of the plane the penalty is ≈ 3.3 , the background costing about a factor of three in $\sigma_{\mathcal{R}}$, but it climbs steeply in a narrow band of nearly pure-proton couplings, reaching 19 for $\mathcal{R} = 0$, $f = 200$ Hz.

The penalty stays within a small factor even where the background exceeds the readout floor by orders of magnitude in power because the background is multiplied by the misalignment factor $M \equiv |1/s_K - 1/s_{Rb}|^2$ from Equation (12). The background displaces both channels alike, along $(1,1)$; the component of that displacement along the signal direction s imitates the drive $\mathcal{N}$, which Equation (11) already profiles away, so only the transverse remainder costs information—and M measures exactly that.⁷ Over most of the band the misalignment is small because both alkalis are largely controlled by the *same* driven ^3He magnetization (Section 2), so s_K/s_{Rb} is insensitive to the coupling structure and only the unequal Fermi-contact factors survive—a $\approx 10\%$ fractional mismatch between the two responses at 100 Hz. The cancellation fails where the ^3He route does not dominate the dynamics: at high frequency, and as $\mathcal{R} \rightarrow 0$, since a proton coupling is barely driving the mostly neutron ^3He spin. This explains the high-penalty region in panel (c).

4.3. Phase and Amplitude Split the Information

The information $I_{\mathcal{R}}^x(S_B)$ of Equation (12) can be split into two terms via $|\partial_{\mathcal{R}} \mathcal{Z}|^2 = (\text{Re } \partial_{\mathcal{R}} \mathcal{Z})^2 + (\text{Im } \partial_{\mathcal{R}} \mathcal{Z})^2$, with the real part the sensitivity of the amplitude ratio $\ln |T_K^b/T_{Rb}^b|$ and the imaginary part that of the differential phase $\Delta\varphi$ of Equation (3). In the small-noise limit (in which the linearized quadratures of $\mathcal{Z}$ are independent Gaussians of common variance $w_{\text{eff}}/|\mathcal{N}|^2$, the phase does not wrap, and the Cramér–Rao bound is attained),⁸ the two terms are the information held separately by phase and amplitude (Appendix A):

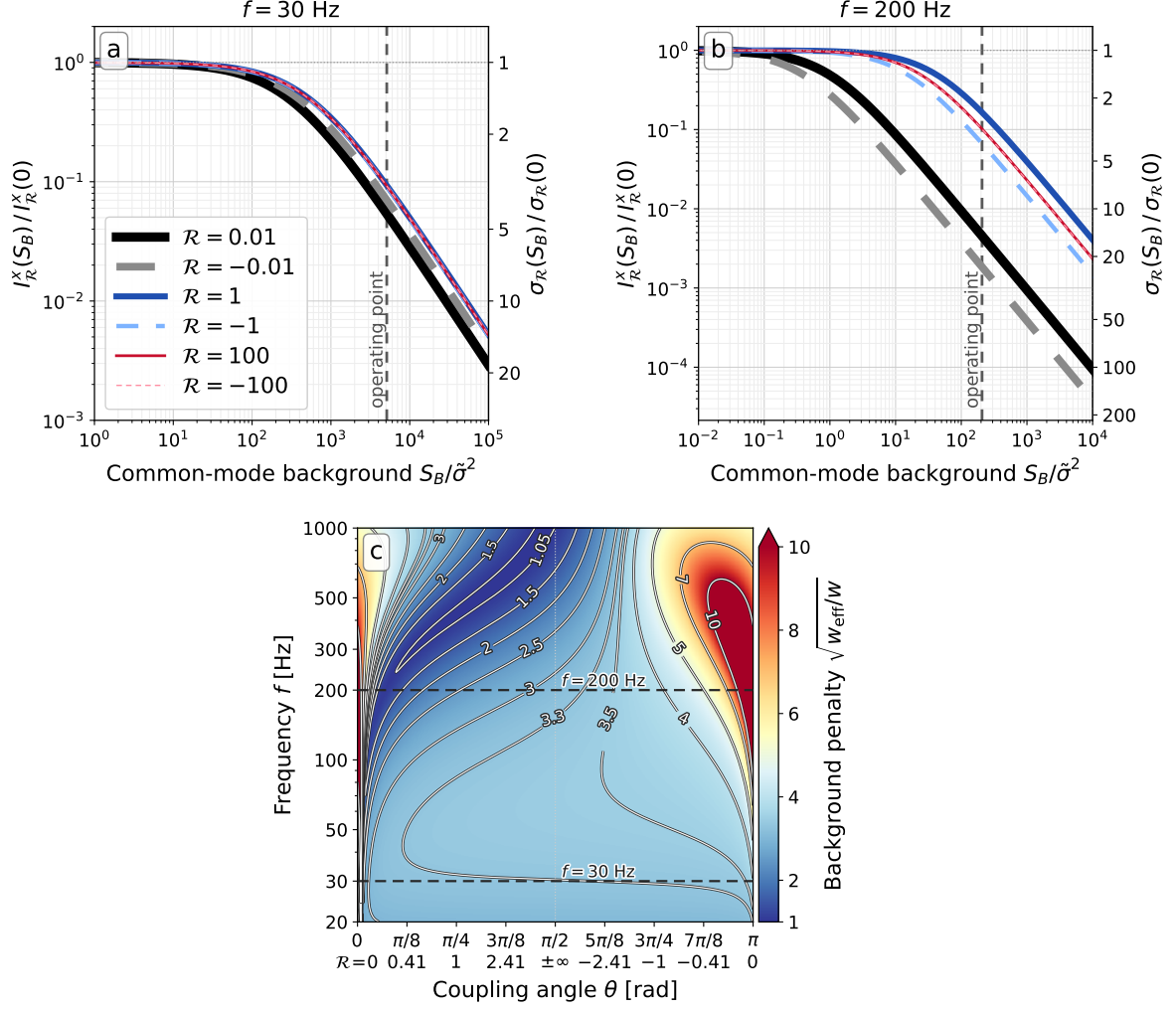


Figure 2. The correlated magnetic background cost in coupling-ratio precision. **(a,b)** The cost as a function of the background size at 30 and 200 Hz. The left axis gives the Fisher information for $\mathcal{R}$ from Equation (11), normalized to the uncorrelated noise optimum, versus the relative common-mode magnetic background $S_B/\tilde{\sigma}^2$; the right axis gives the expected precision on $\mathcal{R}$, normalized the same way, $\sigma_{\mathcal{R}}(S_B)/\sigma_{\mathcal{R}}(0)$. Six coupling structures are shown, from proton-dominated ($\mathcal{R} = \pm 0.01$) through an equal mixture ($\mathcal{R} = \pm 1$) to neutron-dominated ($\mathcal{R} = \pm 100$); solid/dashed marks the sign of $\mathcal{R}$. The vertical line marks the experiment's operating point at that frequency based on the numbers from ref. [18]. **(c)** The background penalty $\sqrt{w_{\text{eff}}/w} = \sqrt{I_{\mathcal{R}}^x(0)/I_{\mathcal{R}}^x(S_B)}$ at the experiment's operating point as a function of the coupling angle θ ($\mathcal{R} = \tan \theta$; $\mathcal{R}$ -value ticks accompany the θ ticks) and frequency. Constant-penalty contours are drawn in white; the dashed horizontal lines indicate the frequencies of panels **(a,b)**.

$$I_{\mathcal{R}}^x(S_B) = I_{\mathcal{R}}^{\Delta\varphi}(S_B) + I_{\mathcal{R}}^{\text{amp}}(S_B),$$

$$I_{\mathcal{R}}^{\Delta\varphi}(S_B) = \frac{(\text{Im } \partial_{\mathcal{R}} \mathcal{Z})^2}{w_{\text{eff}}}, \quad I_{\mathcal{R}}^{\text{amp}}(S_B) = \frac{(\text{Re } \partial_{\mathcal{R}} \mathcal{Z})^2}{w_{\text{eff}}}. \quad (13)$$

Both the differential phase and the amplitude ratio are invariant under the common drive $\mathcal{N}$, which cancels in any inter-channel ratio. They are *not*, however, on the same footing with respect to the *instrument*. The measured ratio is

$$\frac{z_K}{z_{\text{Rb}}} = \frac{\mathcal{K}_K}{\mathcal{K}_{\text{Rb}}} \frac{T_K^b}{T_{\text{Rb}}^b} + \text{noise}, \quad (14)$$

in which the real, positive gain ratio $\mathcal{K}_K/\mathcal{K}_{Rb}$ multiplies the amplitude but leaves the phase untouched. The differential phase $\Delta\varphi$ is therefore immune to *both* the common drive and the channel gains—genuinely calibration free—whereas the amplitude ratio is biased by any error in the gain ratio. In this sense, $\Delta\varphi$ is the calibration-free part of the information.

5. Results

Measurement precision is inversely proportional to the SNR, while the remaining factors that determine how well a quantity (here the coupling angle θ) can be measured are encoded in what we call the *intrinsic precision* κ_θ :

$$\sigma_\theta(S_B) = \frac{\kappa_\theta(S_B)}{\text{SNR}(S_B)}. \quad (15)$$

Apart from its dependence on the correlated magnetic background variance S_B , the intrinsic precision is a function of the frequency and θ . The SNR is the matched-filter combination of the two field-referred channels,

$$\text{SNR}^2(S_B) = |\mathcal{N}|^2 s^\dagger C^{-1} s, \quad (16)$$

with $s_j = T_j^b/T_j^B$ and $C = C(S_B)$ the covariance of Equation (10), reducing at $S_B = 0$ to the sum of the squared single-channel signal-to-noise ratios,

$$\text{SNR}^2(0) = |\mathcal{N}|^2 \sum_j \left(\frac{\mathcal{K}_j |T_j^b|}{\sigma_j} \right)^2. \quad (17)$$

The attainable precision in the measurement of θ is given by

$$\sigma_\theta(S_B) = \sigma_{\mathcal{R}}(S_B) \left| \frac{d\theta}{d\mathcal{R}} \right|, \quad \sigma_{\mathcal{R}}(S_B) = \frac{1}{|\mathcal{N}| \sqrt{I_{\mathcal{R}}^x(S_B)}}, \quad (18)$$

where we used the Cramér–Rao bound, $I_{\mathcal{R}}^x(S_B)$ is given by Equation (12), and $d\theta/d\mathcal{R} = \cos^2 \theta$. The unknown drive amplitude $|\mathcal{N}|$ cancels identically in κ_θ since $\text{SNR} \propto |\mathcal{N}|$ while $\sigma_\theta \propto 1/|\mathcal{N}|$, so the intrinsic precision is a property of the transfer functions and the background alone.

Panel (a) of Figure 3 maps the achievable SNR per unit drive at the operating background, $\text{SNR}(S_B)/B_{\text{eq}} = \gamma_e \sqrt{s^\dagger C^{-1} s}$, the drive being referred to an equivalent field $B_{\text{eq}} = |\mathcal{N}|/\gamma_e$. It is largest for neutron-dominated couplings across most of the band and falls by up to a factor of ~ 30 toward $\theta = 0, \pi$, a deficit deepest at low frequency. The response strength reflects a competition between two drive routes of opposite and frequency-independent preference. The ^3He spin is essentially the neutron’s, so the ^3He -mediated route favors neutron couplings by a factor of 32. The proton spin fractions of ^{39}K and ^{87}Rb carry opposite signs, so the direct alkali route drives the two channels differentially. The inverse-noise weighting suppresses the common mode, in which the near-common-mode neutron drive largely sits, and leaves the differential component at full weight, favoring proton couplings by 4.7 at 200 Hz, a preference flat to 2% across the band. At the bottom of the band the ^3He route carries the alkali signal almost alone and the deficit approaches its full factor of 32; as the ^3He share falls with frequency, the alkali route’s proton preference takes over near 450 Hz, above which proton-dominated couplings reach the larger SNR.

Panel (b) maps the calibration-free readout, the inter-species differential phase $\Delta\varphi = \arg(T_K^b/T_{Rb}^b)$. Panels (c,d) show the intrinsic precision provided by that readout, $\kappa_\theta^{\Delta\varphi} = \sigma_\theta^{\Delta\varphi}(S_B) \cdot \text{SNR}(S_B)$, and the intrinsic precision κ_θ provided by the full two-channel measurement, with $I_\theta^x(S_B) = I_{\mathcal{R}}^x(S_B)/\cos^4 \theta$. The two are nearly indistinguishable above ~ 150 Hz; below that, $\kappa_\theta^{\Delta\varphi}$ diverges in places where $\partial_\theta \Delta\varphi$ vanishes and only the amplitude ratio still responds to the coupling. They share their coupling dependence: below ~ 450 Hz proton-dominated couplings have better intrinsic precision, needing 21 times less SNR at 200 Hz than neutron-dominated ones for the same precision on θ , and the ordering reverses above that frequency. Since $s_j(\theta) = a_j \cos \theta + b_j \sin \theta$, varying θ only rotates the pair $(s, \partial_\theta s)$

Figure 3. Extraction of the coupling ratio $\mathcal{R} = \xi_n / \xi_p$. Each panel maps the 20–1000 Hz band against the coupling angle θ ($\mathcal{R} = \tan \theta$; $\mathcal{R}$ -value ticks accompany the θ ticks). **(a)** Achievable amplitude SNR per unit drive, the drive referred to an equivalent field $B_{\text{eq}} = |\mathcal{N}| / \gamma_e \cdot \text{SNR}(S_B) / B_{\text{eq}} = \gamma_e \sqrt{s^\dagger C^{-1} s}$. **(b)** The calibration-free readout: the inter-species differential phase $\Delta\varphi = \arg(T_K^b / T_{\text{Rb}}^b)$ (logarithmic color scale, linear for $|\Delta\varphi| < 10^{-2}$ rad). **(c)** Intrinsic precision provided by that readout, $\kappa_\theta^{\Delta\varphi} = \sigma_\theta^{\Delta\varphi}(S_B) \cdot \text{SNR}(S_B)$, with $\sigma_\theta^{\Delta\varphi}$ the Cramér–Rao bound from the differential phase alone. **(d)** Intrinsic precision provided by the full two-channel measurement, $\kappa_\theta = \sigma_\theta(S_B) \cdot \text{SNR}(S_B)$. **(e)** Achievable precision on the coupling angle from the differential phase alone, per unit drive: $\sigma_\theta^{\Delta\varphi}(S_B) B_{\text{eq}} = 1 / (\gamma_e \sqrt{I_\theta^{\Delta\varphi}(S_B)})$ with $I_\theta^{\Delta\varphi}(S_B) = I_{\mathcal{R}}^{\Delta\varphi}(S_B) / \cos^4 \theta$, the ratio of panels **(c)** and **(a)**. **(f)** Differential-phase sufficiency $I_\theta^{\Delta\varphi}(S_B) / I_\theta^x(S_B)$, the percentage of the variance-optimal complex-ratio information retained by $\Delta\varphi$ alone. The per-unit-drive maps of panels **(a,e)** are referred to a 1 Hz band. Panels **(a,c–e)** are evaluated at the operating-point background; panels **(b,f)** are fixed by the transfer functions alone. Panels **(c,e,f)** rest on the phase/amplitude split of Equation (13) and inherit its small-noise restriction, while panels **(a,b,d)** do not.

within the fixed field-referred proton and neutron responses a and b , which leaves unchanged the area $\mathcal{A}$ that pair spans in the inverse-noise metric C^{-1} , a function of the frequency alone. The intrinsic precision then depends on the response strength as

$$\kappa_\theta = \frac{s^\dagger C^{-1} s}{\mathcal{A}} = \frac{1}{\gamma_e^2 \mathcal{A}} \left[\frac{\text{SNR}(S_B)}{B_{\text{eq}}} \right]^2, \quad \mathcal{A} = \frac{|a_K b_{\text{Rb}} - a_{\text{Rb}} b_K|}{\sqrt{\det C}}. \quad (19)$$

Therefore, at any given frequency, the couplings with the strongest response have the worst intrinsic precision.

Panel (e) presents the attainable precision on θ from the $\Delta\varphi$ measurement, the ratio of panels (c) and (a), $\sigma_\theta^{\Delta\varphi}(S_B) B_{\text{eq}} = \kappa_\theta^{\Delta\varphi} / [\text{SNR}(S_B) / B_{\text{eq}}]$. Panel (f) quantifies how much of the full coupling-ratio information the calibration-free $\Delta\varphi$ alone captures, through the *differential-phase sufficiency* $I_\theta^{\Delta\varphi}(S_B) / I_\theta^x(S_B) = \sin^2(\arg \partial_\theta \mathcal{Z})$.

Above ~ 100 Hz, $\Delta\varphi$ is near-sufficient over most of the coupling range. At lower frequencies, the sufficiency degrades sharply—the retained fraction falls below 25% near 40 Hz and recovering it would sharpen the precision on θ by the factor $[I_\theta^x(S_B) / I_\theta^{\Delta\varphi}(S_B)]^{1/2} \gtrsim 2$, and at the bottom of the band $\Delta\varphi$ retains only about one percent of the information. The two regimes differ in what drives the channels. Below ~ 100 Hz, both are driven predominantly through the same quasi-static ^3He magnetization and respond nearly in phase, so a change in the coupling is expressed almost entirely in the ratio of the two amplitudes. At higher frequencies, the ^3He roll-off and the alkali antiresonances of Figure 1 separate the two channels' phase lags, and the coupling sweeps the differential phase instead.

Panels (c,e,f) are valid in the small-noise regime of Note 8. Monte Carlo simulation on proton-dominated couplings shows that the full profiled maximum-likelihood estimator and the one based on $\Delta\varphi$ alone attain their Cramér–Rao bounds to within 10% when $\varepsilon \leq 0.3$ and the slope $\partial_\theta \Delta\varphi$ changes by less than 30% over one standard deviation of the $\Delta\varphi$ -based estimate of θ . These conditions set a floor on the detection SNR. Above ~ 150 Hz the floor is typically of order ten, and ≈ 80 on the proton-dominated couplings of best intrinsic precision at 200 Hz, an SNR at which already $\sigma_\theta \lesssim 0.01$ rad. Toward lower frequencies it grows, to ≈ 330 at the ^{87}Rb antiresonance at $\mathcal{R} = 0$ and, above 100 Hz, higher only near a null of the ^{87}Rb exotic response at 90 Hz and $\mathcal{R} \approx 0.016$. Below 100 Hz it typically exceeds 100 and diverges at that null and where $\partial_\theta \Delta\varphi$ vanishes, as $\kappa_\theta^{\Delta\varphi}$ does. At a detection SNR below the floor, panels (c) and (e) give Cramér–Rao bounds rather than attained precisions.

The frequencies quoted in this section are specific to the operating point of Table 2. Varying α over 0.46–0.69 ($\pm 20\%$ around the baseline 0.578) shifts the ^{39}K antiresonance between 195 and 111 Hz, the ~ 100 Hz near-sufficiency threshold between 126 and 67 Hz, and the ~ 450 Hz reversal between 467 and 437 Hz; the two-band structure of Figure 3 persists throughout.

6. Operating-Point and Instrumental Tolerances, and Their Control

The previous sections assume that the transfer functions are known and that the measured differential phase is free of instrumental offsets. We define the tolerance on each systematic effect as the size at which the error $\delta\Delta\varphi$ it causes in the differential phase equals the statistical uncertainty $\sigma_{\Delta\varphi}$ of that phase, the quantity ε of Note 8. By Equation (16), $\sigma_{\Delta\varphi} = \sqrt{w_{\text{eff}} s^\dagger C^{-1} s} / \text{SNR}$, so every tolerance scales as $1/\text{SNR}$. We quote each at a reference SNR = 100, comparable with the small-noise floor on the couplings of best intrinsic precision at 200 Hz, as the tightest value over the frequency–coupling plane unless a narrower domain is stated. Where a tolerance is tightest, $\sigma_{\Delta\varphi}$ is at the milliradian level; elsewhere at SNR = 100 it reaches 0.23 rad on those couplings and, outside the small-noise regime, ≈ 1 rad at the $\mathcal{R} = 0$ ^{87}Rb antiresonance, where the tolerances are correspondingly looser.

6.1. Operating Point

The antiresonance frequencies depend on the longitudinal compensation field B_z as strongly as on $\mathcal{R}$. A 0.5% change in B_z , ≈ 1 nT at the operating point of Table 2, mimics $\delta\mathcal{R} = 0.01$ (Figure 1), so a drift of the operating point would be misread as a change in the coupling ratio. The two dependences

can nevertheless be separated. Since $\mathcal{R}$ enters the Bloch equations only through the exotic couplings of Equation (1), the magnetic transfer functions T_j^B do not depend on $\mathcal{R}$, and the magnetic response determines the operating point in situ, independently of any candidate signal. The compensation field must be held to within 4×10^{-4} of its value, ≈ 80 pT, at the reference SNR.

The limiting drift is expected to be that of the compensation point, which follows the slowly varying ^3He magnetization. It can be corrected by re-setting B_z against the atoms, continuously [31] or periodically [32,33]. A transverse field is briefly modulated (for 12 s in ref. [32]), and the alkali response to it, recorded as a function of B_z , serves as the error signal. This response is a dispersive resonance [31]; ref. [33] locates its zero crossing to 0.05 nT within a minute. The zero crossing, which marks the ^3He field, differs from the compensation point by a set-point offset, so B_z is re-set to it plus that offset [32,33]. In a dual-alkali cell the modulation frequency must lie well above the ^3He Larmor frequency, 6.6 Hz, so that the ^3He barely follows it, and well below the alkali–alkali spin-exchange rate, $\approx 5 \times 10^4$ Hz, so that the alkalis remain locked and present a single zero crossing for the pair. A frequency of a few hundred hertz satisfies both conditions.

The requirement then concerns the drift of B_z between re-settings rather than its absolute stability. If the ^3He magnetization drifts at the rate of the response drifts reported in ref. [32], $\leq 0.02\%$ per hour, the ^3He field moves by ≈ 40 pT per hour, a drift that the hourly re-setting practiced there keeps within the tolerance at the reference SNR; at higher SNR the interval shortens as $1/\text{SNR}$.

At this operating point the model of Section 2 puts the set-point offset at ≈ 3 nT, of which 2 nT come from the alkali magnetization terms of Equation (6) of ref. [18] and the rest from the partial response of the ^3He to the modulation. Refs. [31–33] could neglect this offset or fix it once. Here it exceeds the tolerance by more than an order of magnitude and must be known to a relative accuracy of $\approx 2.5\%$. It follows mainly the alkali magnetization, which, to judge from the response drifts of ref. [32], changes by only a few percent over 20 days, so a daily re-calibration suffices.

The offset is calibrated with the synthetic field of ref. [17], a coil field combined with light-induced fictitious fields on individual alkalis. Applied far below the detection band, it also yields the light-shift difference between the two alkalis, which a single setting of B_z cannot cancel. Since a fractional offset of 10^{-5} in B_z rotates the differential phase of the magnetic response at 0.05 Hz by 63 mrad, this low-frequency measurement resolves the compensation point of each channel well within the tolerance. With an hourly re-setting against the ^3He field and a daily calibration of the offset, B_z stays within its tolerance, and a drift of the operating point cannot be mistaken for a change in the coupling ratio.

6.2. Model Parameters

The remaining parameters of Table 2 bias $\mathcal{R}$ in the same way, but more weakly. Their tolerances at the reference SNR range from 2×10^{-3} on the ^3He density and polarization to 2×10^{-2} on the ^{39}K density, all at least five times looser than that on B_z . They presume that B_z is re-tuned to the compensation point after any change of these parameters, as the re-setting against the ^3He field ensures. The same hourly re-setting meets the tightest of them, on the ^3He magnetization, since it locates the ^3He field to a few parts in 10^4 [33]. The alkali densities and relaxation rates follow the cell temperature at $\approx 5\%$ per kelvin, and comagnetometer ovens hold the temperature to $\lesssim 0.1^\circ\text{C}$ [34,35], which corresponds to half a percent in density; the percent-level tolerances depend on the oven, and any tighter ones on re-calibration.

Swept across frequency, the synthetic field re-determines these parameters and the alkali polarizations. Its fictitious fields also reach the differential alkali mode, which a magnetic field barely excites. With response drifts of $\leq 0.02\%$ per hour [32], the daily sweep that accompanies the offset calibration covers the ^{39}K density with a fourfold margin. The only combination the sweep leaves undetermined, n_{He} and P_z^{He} separately at fixed magnetization, is one to which the inferred $\mathcal{R}$ is also insensitive.

6.3. Probe Cross-Talk

Because each probe also senses the dispersive tail of the other alkali, each channel contains a fraction χ of the other channel [18]. The effect could be enhanced near the antiresonances, where one

alkali's response is small while the other's is not. To first order, the admixture shifts the log-ratio $\mathcal{Z}$ by $\delta\mathcal{Z} = \chi (T_{\text{Rb}}^b/T_{\text{K}}^b - T_{\text{K}}^b/T_{\text{Rb}}^b)$. Equal fractions in the two channels are the worst case, because the two admixtures then enter $\text{Im } \delta\mathcal{Z}$ with the same sign. At $\mathcal{R} = 0$ the antiresonances are shallow, with $0.6 \lesssim |T_{\text{K}}^b/T_{\text{Rb}}^b| \lesssim 2.6$ across the band, and at a deep antiresonance the shift and $\sqrt{w_{\text{eff}}}$ grow at the same rate, leaving the tolerance unchanged. The resulting tolerance at the reference SNR is $\chi \lesssim 10^{-2}$, tightest at the lower band edge.

The admixture need not be measured, since it is fixed by the probe detunings and the alkali density ratio through Equation (1) of ref. [18]. For the probes of that work it is $\approx -0.8\%$ on the ^{39}K channel and $\approx -0.1\%$ on the ^{87}Rb channel, the percent level stated there. Once these values are entered in the model, only their uncertainty biases $\mathcal{R}$; a 10% error in the density ratio leaves a residual admixture of 10^{-3} , an order of magnitude below the tolerance. Cross-talk is therefore negligible, near the antiresonances or elsewhere, except in the immediate vicinity of the null of the ^{87}Rb exotic response identified in Section 5, where $|T_{\text{K}}^b/T_{\text{Rb}}^b|$ exceeds 100. There, at $\chi = 10^{-2}$, the admixture exceeds the ^{87}Rb response it contaminates, the expansion in χ fails, and the null is not observable.

6.4. Differential Chain Phase

The calibration-free property presumes real, positive gains $\mathcal{K}_j$. A phase difference $\varphi_d \equiv \arg(\mathcal{K}_{\text{K}}/\mathcal{K}_{\text{Rb}})$ between the two detection chains adds to $\Delta\varphi$ and has the same tolerance, $\sigma_{\Delta\varphi}$. At the reference SNR this is 0.5 mrad at the most demanding point, the narrow near-common-mode trough of panel (a) of Figure 3 at $\mathcal{R} \approx 0.06$ near 80 Hz, and 3.7 mrad at $\mathcal{R} = 1$ over 100–200 Hz.

The static part of φ_d , of order 0.1 mrad at 80 Hz for photodiode amplifiers of 100 kHz bandwidth matched to 10%, is measured and removed by the calibration described below; the shared lock-in of ref. [18] contributes nothing to it. The tolerance therefore constrains only the varying part, namely the drift of the electronics between calibrations and the dependence of each chain's phase φ_j on its signal amplitude $|z_j|$ at the demodulator. For a conversion coefficient $\zeta_j \equiv \partial\varphi_j/\partial \ln |z_j|$ with $|\zeta_j| \lesssim 0.05$ rad, the 0.5 mrad bound is met by holding the probe power of each channel, which scales $|z_j|$, stable to 0.5%, a level that standard intensity stabilization provides [33–35].

The sweep of Section 6.2, which necessarily spans the band, measures φ_d in situ against the magnetic response at each frequency. Applied at several amplitudes spanning those of a detectable signal, with each probe's power scaled in turn, it also measures ζ_j . The differential phase of the magnetic response, against which φ_d is measured, can be computed in advance and is insensitive to residual errors in the operating point and parameters; with these at their tolerances, it moves by less than 0.25 mrad across the band, which is below the tolerance on φ_d . Only the value of φ_d at the signal frequency enters $\Delta\varphi$, and a candidate is followed by a dedicated measurement at its frequency. In that measurement, the synthetic field, set to mimic the exotic drive for trial values of $\mathcal{R}$, measures $\Delta\varphi(\mathcal{R})$ directly, including the chain phase.

7. Summary and Conclusions

Ref. [18] proposed the inter-species phase difference $\Delta\varphi$ of a single-cell dual-alkali comagnetometer as a calibration-free readout of the coupling ratio $\mathcal{R} = \xi_n/\xi_p$, which the two alkalis encode in their *relative* response because they carry different mixtures of the neutron and proton couplings.

Treating the extraction of $\mathcal{R}$ as a statistical estimation problem, we have shown here that profiling the common drive out of the two-channel likelihood leaves the maximum-likelihood estimate of $\mathcal{R}$ depending on the two channels only through their complex ratio, with Fisher information $|\partial_{\mathcal{R}}\mathcal{Z}|^2/w_{\text{eff}}$ at any SNR, that $\mathcal{Z}$ splits into $\Delta\varphi$ and an amplitude ratio, and that only $\Delta\varphi$ is immune to the inter-channel gain. The measured ratio itself carries the full information only in the small-noise regime.

We mapped, over the 20–1000 Hz band and the full range of coupling angles, the differential phase together with the quantities that govern its use: the attainable detection SNR per unit drive, the intrinsic precision on the coupling angle provided by $\Delta\varphi$ and by the full two-channel measurement, the overall precision that $\Delta\varphi$ affords at a given drive, and the share of the coupling information it retains. At any given frequency the couplings with the strongest response have the worst intrinsic

precision. Below ~ 450 Hz the neutron-dominated couplings reach the larger SNR while the proton-dominated ones have better intrinsic precision, and above it both statements reverse. We also found that above ~ 100 Hz, $\Delta\varphi$ is near-sufficient over most of the coupling range and is therefore the natural and robust readout for the coupling ratio. At lower frequencies the amplitude ratio holds a substantial part of the information, recoverable only at the price of a calibrated inter-channel gain. Finally, we derived the tolerances on the operating point and on the instrumental effects that could bias $\Delta\varphi$, and showed that they can be met in situ.

The same $\mathcal{Z}$ ratio construction that removes the unknown drive also removes any information about the size of the signal, so the extraction of $\mathcal{R}$ and the detection of the field are complementary uses of the same two channels, the latter resting on the amplitude combination that cancels the magnetic background [18]. Within the structure established here, no more elaborate observable can improve upon $\Delta\varphi$ without forfeiting the calibration-free property, the remaining information residing entirely in the amplitude ratio. A single-cell dual-alkali comagnetometer thus offers, beyond its sensitivity as a field detector, an intrinsic calibration-free handle on the *internal structure* of a candidate dark matter coupling.

Author Contributions: Conceptualization, Y.R., Y.K. and E.S.; methodology, Y.R. and Y.K.; software, Y.R., Y.K. and Y.J.; validation, Y.R. and Y.K.; formal analysis, Y.R., Y.K. and Y.J.; writing—original draft preparation, Y.K.; writing—review and editing, Y.R., E.S., Y.J. and R.F.; visualization, Y.K.; supervision, Y.K., E.S. and R.F.; project administration, Y.R., Y.K., E.S. and R.F.; funding acquisition, Y.K., E.S., M.G. and R.F. All authors have read and agreed to the published version of the manuscript.

Funding: We gratefully acknowledge support from the Gordon and Betty Moore Foundation, Simons Foundation, Alfred P. Sloan Foundation, and John Templeton Foundation. This work was supported in part also by the United States–Israel Binational Science Foundation Grants No. 2016635 and No. 2018257 and the Israel Innovation Authority Grants No. 67082 and No. 74482. YK is supported in part by the Israel Science Foundation Grant No. 1666/22.

Data Availability Statement: The data presented in this study are available on request from the corresponding author.

Acknowledgments: During the preparation of this manuscript, the authors used Anthropic’s Claude models (Opus 4.8, Opus 5, and Fable 5) for the purposes of text drafting and editing, figure generation, and numerical analysis and cross-checking. The authors have reviewed and edited the output and take full responsibility for the content of this publication.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Reduction of the Profiled Information and the Phase/Amplitude Split

This appendix proves that the profiled two-channel Fisher information of Equation (11) reduces to the closed form of Equation (12), and that in the small-noise limit its two quadratures carry the phase/amplitude split of Equation (13). Throughout, $s = (s_K, s_{Rb})^\top$ with $s_j = T_j^b/T_j^b$ is the field-referred response of Section 4, $\partial_{\mathcal{R}}s$ its coupling-ratio derivative, and C the covariance of Equation (10); we abbreviate the inverse-noise inner product $\langle p, q \rangle \equiv p^\dagger C^{-1} q$. In this notation, Equation (11) is the ratio of the Gram determinant of s and $\partial_{\mathcal{R}}s$ to the squared norm $\langle s, s \rangle$:

$$I_{\mathcal{R}}^x(S_B) = \langle \partial_{\mathcal{R}}s, \partial_{\mathcal{R}}s \rangle - \frac{|\langle s, \partial_{\mathcal{R}}s \rangle|^2}{\langle s, s \rangle} = \frac{\det G}{\langle s, s \rangle}, \quad G = \begin{pmatrix} \langle s, s \rangle & \langle s, \partial_{\mathcal{R}}s \rangle \\ \langle \partial_{\mathcal{R}}s, s \rangle & \langle \partial_{\mathcal{R}}s, \partial_{\mathcal{R}}s \rangle \end{pmatrix}. \quad (A1)$$

Let $J = [s \mid \partial_{\mathcal{R}}s]$, so that $G = J^\dagger C^{-1} J$. By multiplicativity of the determinant, $\det G = \det J \det(C^{-1}) \det J = |\det J|^2 / \det C$; what remains is to evaluate $\det J$ and $\langle s, s \rangle$.

The determinant of J is

$$\det J = s_K \partial_{\mathcal{R}}s_{Rb} - s_{Rb} \partial_{\mathcal{R}}s_K = -s_K s_{Rb} \left(\frac{\partial_{\mathcal{R}}s_K}{s_K} - \frac{\partial_{\mathcal{R}}s_{Rb}}{s_{Rb}} \right) = -s_K s_{Rb} \partial_{\mathcal{R}}\mathcal{Z}, \quad (A2)$$

where $\partial_{\mathcal{R}} \mathcal{Z} = \partial_{\mathcal{R}} \ln(s_K/s_{Rb}) = \partial_{\mathcal{R}} \ln(T_K^b/T_{Rb}^b)$, the two logarithms differing only by the $\mathcal{R}$ -independent $\ln(T_K^B/T_{Rb}^B)$. Hence $|\det J|^2 = |s_K s_{Rb}|^2 |\partial_{\mathcal{R}} \mathcal{Z}|^2$.

Since $(\det C) C^{-1} = \text{adj } C$, $\det C \langle s, s \rangle = s^\dagger (\text{adj } C) s$. For the real symmetric C of Equation (10), $\text{adj } C = \begin{pmatrix} \tilde{\sigma}_{Rb}^2 + S_B & -S_B \\ -S_B & \tilde{\sigma}_K^2 + S_B \end{pmatrix}$, and

$$s^\dagger (\text{adj } C) s = \tilde{\sigma}_{Rb}^2 |s_K|^2 + \tilde{\sigma}_K^2 |s_{Rb}|^2 + S_B |s_K - s_{Rb}|^2. \quad (\text{A3})$$

Dividing by $|s_K s_{Rb}|^2$,

$$\frac{s^\dagger (\text{adj } C) s}{|s_K s_{Rb}|^2} = \underbrace{\frac{\tilde{\sigma}_K^2}{|s_K|^2} + \frac{\tilde{\sigma}_{Rb}^2}{|s_{Rb}|^2}}_w + S_B \left| \frac{1}{s_K} - \frac{1}{s_{Rb}} \right|^2 = w_{\text{eff}}, \quad (\text{A4})$$

the first two terms being w of Equation (7) (in field-referred form, $\tilde{\sigma}_j/|s_j| = \sigma_j/(\mathcal{K}_j|T_j^b|)$).

Combining Equations (A1)–(A4) and using $\det C \langle s, s \rangle = s^\dagger (\text{adj } C) s$, we find that the common factor $|s_K s_{Rb}|^2$ cancels:

$$I_{\mathcal{R}}^x(S_B) = \frac{\det G}{\langle s, s \rangle} = \frac{|\det J|^2}{s^\dagger (\text{adj } C) s} = \frac{|\partial_{\mathcal{R}} \mathcal{Z}|^2}{w_{\text{eff}}}, \quad (\text{A5})$$

which is Equation (12). At $S_B = 0$, $w_{\text{eff}} = w$ and Equation (A5) reduces to Equation (8); the Lagrange identity quoted in Note 4 in Section 4 is the special case $C = \text{diag}(\tilde{\sigma}_K^2, \tilde{\sigma}_{Rb}^2)$.

The reduction above uses the specific form of Equation (10) only in Equation (A3). For any Hermitian C , $w_{\text{eff}} = s^\dagger (\text{adj } C) s / |s_K s_{Rb}|^2$, so a disturbance of density S entering the channels with complex weights (c_K, c_{Rb}) adds $S |c_K/s_K - c_{Rb}/s_{Rb}|^2$ to w_{eff} , of which Equation (12) is the case $c = (1, 1)$. Only the part of such a disturbance that is not proportional to the signal direction s costs information.

The split of Equation (13) follows from the same w_{eff} , read now as a variance. Linearizing the field-referred log-ratio $\ln(x_K/x_{Rb})$ about its noiseless value $\ln(s_K/s_{Rb})$ gives a fluctuation $\delta \mathcal{Z}$ with $\mathcal{N} \delta \mathcal{Z} = u^\top (x - \mathcal{N}s)$, where $x - \mathcal{N}s$ is the field-referred noise of Equation (9), of covariance C , and $u = (1/s_K, -1/s_{Rb})^\top$. Being a fixed linear combination of jointly circular Gaussians, $\mathcal{N} \delta \mathcal{Z}$ is itself circular, so its two quadratures are independent Gaussians of the common variance (formed with C itself, not the inverse-noise metric)

$$u^\dagger C u = \frac{\tilde{\sigma}_K^2}{|s_K|^2} + \frac{\tilde{\sigma}_{Rb}^2}{|s_{Rb}|^2} + S_B \left| \frac{1}{s_K} - \frac{1}{s_{Rb}} \right|^2 = w_{\text{eff}}, \quad (\text{A6})$$

which is Equation (A4) once more, reached as a variance rather than as an algebraic combination. The two quadratures are therefore independent Gaussian observations whose means carry the $\mathcal{R}$ -dependence of $\text{Re } \mathcal{Z} = \ln |T_K^b/T_{Rb}^b|$ and $\text{Im } \mathcal{Z} = \Delta \varphi$, the two log-ratios differing only by the $\mathcal{R}$ -independent constant noted below Equation (A2). Their common variance $w_{\text{eff}}/|\mathcal{N}|^2$ depends on $\mathcal{R}$ as well, but only through an overall scale; that scale is degenerate with the profiled drive amplitude and therefore carries no information about $\mathcal{R}$. The Fisher information of such an observation is the squared derivative of its mean divided by that variance, giving $(\text{Re } \partial_{\mathcal{R}} \mathcal{Z})^2/w_{\text{eff}}$ and $(\text{Im } \partial_{\mathcal{R}} \mathcal{Z})^2/w_{\text{eff}}$ per unit $|\mathcal{N}|^2$ —the two terms of Equation (13)—and independent observations contribute additively. Their sum $|\partial_{\mathcal{R}} \mathcal{Z}|^2/w_{\text{eff}}$ is Equation (A5), so the linearized quadratures recover the exact profiled information. The ratio $I_{\mathcal{R}}^{\Delta \varphi}(S_B)/I_{\mathcal{R}}^{\text{amp}}(S_B) = \tan^2(\arg \partial_{\mathcal{R}} \mathcal{Z})$ is independent of w_{eff} : the background S_B rescales the total information but leaves the phase/amplitude division untouched, and with it the differential-phase sufficiency $I_{\mathcal{R}}^{\Delta \varphi}(S_B)/I_{\mathcal{R}}^x(S_B)$.

Notes

- ¹ Coupling to electrons need not be present since quarks and leptons are independent degrees of freedom.
- ² The superscript b refers to the exotic field, which is denoted by $\mathbf{b}$ in ref. [18]; it is to be distinguished from the superscript B of the magnetic transfer function T_j^B introduced below.

- 3 Circularity follows from the demodulation, the shot noise of a linear polarimeter having no preferred phase with respect to the lock-in reference. Independence between the channels follows from the separate probe beams and photodetectors, whose shot noise is uncorrelated. Gaussianity follows from the photon number per demodulated sample: a one-second sample carries more than 10^{12} photons even at a probe power of a microwatt. Whiteness is required only across the detection bandwidth Δf about the signal frequency. The probe modulation of ref. [18] places the detection band above the $1/f$ region of the polarimetry chain, where the spectrum is flat over any Δf of interest.
- 4 Profiling the Gaussian likelihood over $\mathcal{N}$ leaves the Fisher information for $\mathcal{R}$ as the projection of $\partial_{\mathcal{R}}\mu$ off the drive direction $\mu_j \equiv \mathcal{K}_j T_j^b$; for two channels Lagrange's identity collapses it to $|\partial_{\mathcal{R}}\mathcal{Z}|^2/w$; Appendix A carries out the reduction in full, including the correlated-background generalization of Equation (12). Treating $\mathcal{Z}$ itself as Gaussian with variance w (an approximation) happens to lead to the same expression as the exact approach.
- 5 The complete information for $\mathcal{R}$ is $|\mathcal{N}|^2 I_{\mathcal{R}}^{\mathcal{Z}}$, so the drive amplitude will reappear as the explicit factor $|\mathcal{N}|$ in the precision $\sigma_{\mathcal{R}} = 1/(|\mathcal{N}|\sqrt{I_{\mathcal{R}}^{\mathcal{Z}}})$.
- 6 We neglect the spin-projection noise from the alkalis' mutual spin exchange. Correlated spin-projection noise predominantly excites the exchange-locked collective mode and therefore enters the two channels approximately common-mode, like the magnetic background itself: it is subsumed in S_B , a fractional addition of only $\sim 2 \times 10^{-4}$ for the ~ 0.1 fT/ $\sqrt{\text{Hz}}$ budgeted in ref. [18], while the residual non-common part lies well below the readout floors $\tilde{\sigma}_j$ across the band.
- 7 A correlated disturbance entering the two channels with complex weights $c = (c_K, c_{\text{Rb}})$ costs information only through its part not in the ratio $s_K : s_{\text{Rb}}$ (Appendix A); the magnetic background, $c = (1, 1)$, is the case carried by M above. For the ^{87}Rb -selective light-shift disturbance of ref. [18], which departs from common mode by $|1 - c_K/c_{\text{Rb}}| \approx 0.07$ and whose common-mode part is already contained in S_B , the remainder raises $\sigma_{\mathcal{R}}$ by at most 1.5% at the level considered there.
- 8 The small quantity is $\varepsilon \equiv \sqrt{w_{\text{eff}}}/|\mathcal{N}|$, the standard deviation of each quadrature of $\mathcal{Z}$ and hence of the differential phase. Section 5 delimits the small-noise regime, pairing the limit on ε with a companion condition on the linearity of $\Delta\varphi(\theta)$; the first is required for the phase/amplitude split of Equation (13), the second only for attaining the Cramér–Rao bounds. Equation (12) itself is the *exact* drive-profiled Fisher information of the model of Equation (4) at any SNR, so neither the total information nor the intrinsic precision depends on either condition.

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