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Higher-order corrections to mass-charge relation of extremal black holes

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Abstract: We investigate the hypothesis that the higher-derivative corrections always make extremal non-supersymmetric black holes lighter than the classical bound and self-repulsive. This hypothesis was recently formulated in the context of the so-called swampland program. One of our examples involves an extremal heterotic black hole in four dimensions. We also calculate the effect of general four-derivative terms in Maxwell-Einstein theories in D dimensions. The results are consistent with the conjecture.

Keywords: Black Holes in String Theory, Superstring Vacua, Black Holes.

Contents

1. Introduction

In view of the seemingly large number of allowed vacua in string theory, it is important to look for universal properties of these solutions, and see what features of the low-energy field theory can nevertheless be deduced from string theory. It turns out that such features exist, and not any low-energy particle content is allowed: Vafa [1] has discussed the possibility of restrictions related to the finiteness of volume of massless scalar fields, the finiteness of the number of massless fields, and the rank of the gauge groups. In fact, just the requirement to include quantum gravity (even if not in the framework of string theory) puts constraints on the low-energy physics [2 − 11]. Arkani-Hamed et al. [2] considered a theory of a single U(1) gauge field, and came to the conclusion that the gauge force must be stronger than gravity, i.e., there must exist charged particles for which the net force is repulsive. Furthermore, the effective theory breaks down at some scale *beneath* the Planck scale, and there should exist a charged particle at or below that scale.

In particular, Arkani-Hamed et al. made a prediction regarding the mass-charge relation of extremal black holes. Consider a particle with a mass M and a charge Q. For this particle to be unstable, it must be able to decay into two or more particles whose total mass is smaller than M and total charge equal to Q. To satisfy these conditions, at least one of the outgoing particles must have a smaller M/Q ratio than the original particle.

The argument extends to black holes, which are believed to be the low-energy description of elementary particles whose masses are much above the Planck scale. Since it is unnatural to have an infinite number of exactly stable particles, the mass-charge relation for extremal black holes $M = Q$ cannot be exact: the M/Q ratio for extremal black holes should decrease with decreasing Q, so that for every extremal black hole there is another black hole with a smaller M/Q ratio (see figure 1). Because states with $M/Q < 1$ must

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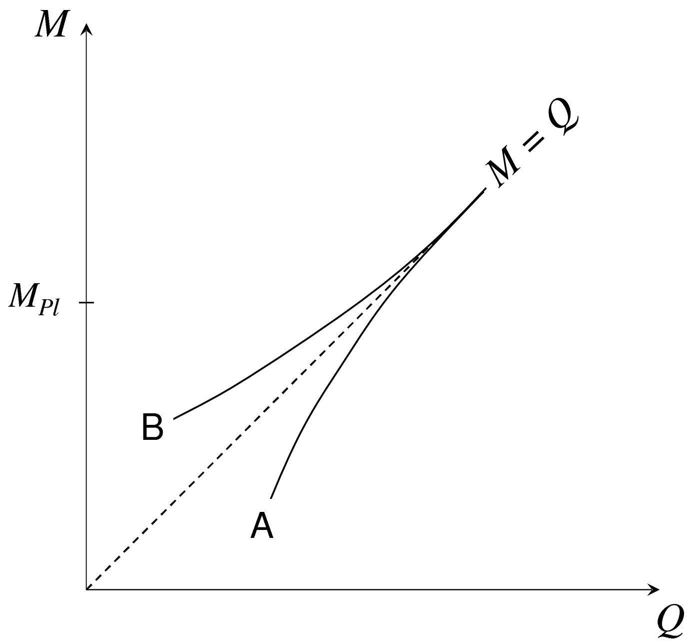

Figure 1: The classical mass-charge relation for extremal black holes is represented by the dashed line; it must be valid in the limit $M \gg M_{\rm Pl}$. Curve A shows a possible exact mass-charge relation. Curve B is unacceptable because it would imply an infinite number of states that cannot decay.

exist, the most natural expectation is that the black holes, states with very high values of M, Q, also satisfy $M/Q < 1$, although the difference from 1 is tiny.

Since the net force between black holes with $M = Q$ vanishes, the previous argument also predicts that the net force will become repulsive. This is indeed expected because if the force were attractive, heavier bound states with a lower M/Q ratio would be possible, again creating an infinite number of states that cannot decay. While the relation between the decrease of the mass and the repulsion is trivial in the case of Reissner-Nordström black holes, the existence of other fields (e.g., the dilaton) makes the two arguments independent.

In this paper we present calculations concerning corrections to the mass-charge relation of extremal black holes. Section 2 is dedicated to the case of four-derivative terms affecting Reissner-Nordström black holes (and appendix B extends the result to the case of D dimensions). Section 3 discusses a heterotic black hole where the additional coupling to the dilaton must be included. In section 4, we offer conclusions and a list of black objects that could be investigated.

2. Corrections to the Reissner-Nordström black hole

The Reissner-Nordström black hole is a spherically symmetric static solution with a radial electric (or magnetic) field, governed by the action:

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{2\kappa^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{2.1}$$

where $\kappa^2 = 8\pi G$. Starting with the most general spherically symmetric static metric

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2 \tag{2.2}$$

and looking for a solution with a radial electric field of the form $F^{01} = E(r)$, one finds

$$e^{\nu(r)} = e^{-\lambda(r)} = 1 - \frac{\kappa^2 M}{4\pi r} + \frac{\kappa^2 Q^2}{32\pi^2 r^2} \qquad E(r) = \frac{Q}{4\pi r^2} \tag{2.3}$$

The solution describes a black hole for $M \geq \frac{\sqrt{2}}{\kappa}|Q|$ (otherwise the solution describes a naked singularity). Black holes with the minimal possible mass M for a given charge Q are called extremal. In units with $\kappa^2 = 2$, they satisfy $M = |Q|$ and the horizon radius $r = M/4\pi = |Q|/4\pi$.

Corrections due to quantum gravity can be represented by higher-order terms in the effective action. For the purpose of determining the mass of an extremal black hole, we are interested in the solution near the horizon: $r \sim Q$. The unperturbed solution (2.3) implies that any derivative contributes a factor of order $1/Q$, so the Riemann tensor is $R \sim 1/Q^2$, and for the electromagnetic field tensor we have $F \sim Q/r^2 \sim 1/Q$ and $\nabla F \sim 1/Q^2$. Since $Q \simeq M \gg 1$, terms of higher order in R, F, and derivatives are suppressed by powers of $1/Q$, and we may consider just the leading-order corrections. Both terms in (2.1) are $\sim 1/Q^2$. The leading order ($\sim 1/Q^4$) corrections are:

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big(&\frac{R}{2\kappa^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_1\,R^2 + c_2\,R_{\mu\nu}R^{\mu\nu} + c_3\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \\ &+ c_4\,RF_{\mu\nu}F^{\mu\nu} + c_5\,R^{\mu\nu}F_{\mu\rho}F_{\nu}{}^{\rho} + c_6\,R^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + c_7\,(F_{\mu\nu}F^{\mu\nu})^2 \\ &+ c_8\,(\nabla_\mu F_{\rho\sigma})(\nabla^\mu F^{\rho\sigma}) + c_9\,(\nabla_\mu F_{\rho\sigma})(\nabla^\rho F^{\mu\sigma})\Big) \end{aligned} \tag{2.4}$$

We did not include a $(\nabla_\mu F^{\mu\nu})(\nabla^\rho F_{\rho\nu})$ term because $(\nabla_\mu F^{\mu\nu})$ and $(\nabla^\rho F_{\rho\nu})$ vanish in the unperturbed solution, so variations of this term are proportional to additional powers of the correction coefficients c_i. A similar argument applies to $\widetilde{c}_7 F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}$, whose contribution to the equations of motion (to first order in c_i) turns out to be equal to half the contribution of $(F^{\mu\nu}F_{\mu\nu})^2$, related to the fact that only F^{01} and F^{10} are non-zero in the unperturbed solution. Therefore, in our problem $\widetilde{c}_7$ can be absorbed in c_7.

The solution of the equations of motion for the metric is straightforward [12]. First, one can note that the spherical symmetry made it possible to express $\lambda(r)$ and $\nu(r)$ explicitly in terms of $R_{\mu\nu}$ as

$$e^{-\lambda} = 1 - \frac{\kappa^2 M}{4\pi r} - \frac{1}{r}\int_r^\infty dr\, r^2 \left(\frac{R_0^0 - R_1^1}{2} - R_2^2\right) \tag{2.5}$$

$$\nu = -\lambda + \int_r^\infty dr\, r\left(R_0^0 - R_1^1\right) e^{\lambda} \tag{2.6}$$

Next, recall Einstein's equation in the form

$$R_{\mu\nu} = \kappa^2\left(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}\right) \qquad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \tag{2.7}$$

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where $T = T_0^0 + T_1^1 + T_2^2 + T_3^3$ (with $T_3^3 = T_2^2$). Then (2.5) and (2.6) become

$$e^{-\lambda} = 1 - \frac{\kappa^2 M}{4\pi r} - \frac{\kappa^2}{r}\int_r^\infty dr\, r^2\, T_0^0 \tag{2.8}$$

$$\nu = -\lambda + \kappa^2 \int_r^\infty dr\, r\left(T_0^0 - T_1^1\right)\, e^\lambda \tag{2.9}$$

We take the higher-order terms in the action (2.4) to be a perturbation, treat them as a part of S_{matter}, and use the unperturbed solution (2.3) to calculate their corresponding $T_{\mu\nu}$. We also vary the action with respect to the gauge field to obtain corrections to Maxwell's equations, which modify the contribution of the $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ term to $T_{\mu\nu}$. The calculation of these two contributions to the effective $T_{\mu\nu}$ is presented in appendix A. The corrected metric in terms of $m = M/4\pi$ and $q = Q/4\pi$ is

$$\begin{aligned} e^{-\lambda} = 1 &- \frac{\kappa^2 m}{r} + \frac{\kappa^2 q^2}{2r^2} + \frac{q^2}{r^6}\left(c_2 \frac{\kappa^4}{5}\left(-6\kappa^2 q^2 + 15 m\kappa^2 r - 20 r^2\right)\right. \\ &+ c_3 \frac{\kappa^4}{5}\left(-24\kappa^2 q^2 + 60\kappa^2 m r - 80 r^2\right) + c_4 \kappa^2\left(-6\kappa^2 q^2 + 14\kappa^2 m r - 16 r^2\right) \\ &+ c_5 \frac{\kappa^2}{5}\left(-11\kappa^2 q^2 + 25\kappa^2 m r - 30 r^2\right) + c_6 \frac{\kappa^2}{5}\left(-16\kappa^2 q^2 + 35\kappa^2 m r - 40 r^2\right) \\ &+ c_7\left(\frac{-4\kappa^2 q^2}{5}\right) + c_8 \frac{\kappa^2}{5}\left(6\kappa^2 q^2 - 15\kappa^2 m r + 20 r^2\right) \\ &\left. + c_9 \frac{\kappa^2}{10}\left(6\kappa^2 q^2 - 15\kappa^2 m r + 20 r^2\right)\right) \end{aligned} \tag{2.10}$$

The mass-charge relation for extremal black holes becomes

$$\frac{\kappa}{\sqrt{2}}\frac{M}{|Q|} = 1 - \frac{2}{5q^2}\left(2c_2 + 8c_3 + \frac{2c_5}{\kappa^2} + \frac{2c_6}{\kappa^2} + \frac{8c_7}{\kappa^4} - \frac{2c_8}{\kappa^2} - \frac{c_9}{\kappa^2}\right) \tag{2.11}$$

Then the conjecture of Arkani-Hamed et al. implies that our low-energy effective theory must satisfy

$$2c_2\kappa^4 + 8c_3\kappa^4 + 2c_5\kappa^2 + 2c_6\kappa^2 + 8c_7 - 2c_8\kappa^2 - c_9\kappa^2 \geq 0 \tag{2.12}$$

We performed the same calculation in D spacetime dimensions, and the results are presented in appendix B.

We can use our results to check whether higher-order terms in the string theory effective action increase or decrease the mass-charge ratio in certain special cases. A U(1) gauge field can arise as a subgroup of the $E_8 \times E_8$ or SO(32) gauge group in the low-energy effective theory of the heterotic string. We would like to consider a black hole charged under this U(1), while we set the remaining gauge fields and the antisymmetric field strength $H_{\mu\nu\rho}$ to zero. Consider heterotic string theory compactified on a $(10 - D)$-dimensional torus. If we are able to stabilize the dilaton, then one possible background is a D-dimensional Reissner-Nordström black hole. (A black hole that involves the dilaton as well is discussed in the next section.) The ten-dimensional Lagrangian is [13]:

$$\begin{aligned} \mathcal{L} = \frac{1}{2\kappa_{10}^2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha' h}{16\kappa_{10}^2}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \\ -\frac{3}{64}\alpha' h \kappa_{10}^2\left((F_{\mu\nu}F^{\mu\nu})^2 - 4F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}\right) \end{aligned} \tag{2.13}$$

The dilaton has been set to a constant ϕ_0 and $h \equiv e^{-\kappa_{10}\phi_0/\sqrt{2}}$. Such an assumption may be physically interpreted as a consequence of a dynamically generated potential for the dilaton in a particular compactification: the dilaton acquires mass much greater than the inverse radius of the black hole, its effects may be neglected, while the terms we consider are preserved. In $D = 4$, the Gauss-Bonnet combination

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

is a topological invariant and does not influence the equations of motion. It does have an effect in other dimensions, where it interestingly cancels the $(3D - 7)$ factor in (B.15). While the effect of the Gauss-Bonnet terms is to increase the mass, the combination of the F^4 terms decreases the mass. (Note also that when the F^4 terms are expressed in terms of $(F^2)^2$ and $(F\widetilde{F})^2$, their coefficients are positive, much like in the Dirac-Born-Infeld action: this fact is required by the energy conditions or, equivalently, the unitarity [14].) With

$$c_1 = c_3 = \frac{h\alpha'}{16\kappa^2} \qquad c_2 = -\frac{h\alpha'}{4\kappa^2} \qquad c_7 = \frac{3h\alpha'\kappa^2}{64} \tag{2.14}$$

where we absorbed $\widetilde{c}_7$ in c_7 as explained after eq. (2.4), we obtain

$$\frac{D-3}{D-2}\frac{\kappa^2 M^2}{Q^2} = 1 - \alpha'\frac{(D-3)(2D-5)h}{4(3D-7)}\left(\frac{(D-2)(D-3)\Omega^2_{D-2}}{\kappa^2 Q^2}\right)^{1/(D-3)} \tag{2.15}$$

The overall effect is to lower M/Q for $D > 3$, as we indeed expect for a theory that includes quantum gravity.

Interestingly, the leading term in D canceled in (2.15), which might be relevant in large-D expansions. The reader may also notice that the leading mass correction parametrically agrees with the relation for perturbative string excitations only in $D = 4$, where both relations can be written as

$$M^2 = aQ^2 - b \tag{2.16}$$

where a and b are constants.

3. Corrections to the GHS black hole

In general, the low-energy effective action of the heterotic string includes also the dilaton field ϕ, which is sourced by the gauge field:

$$S = \int d^4x\sqrt{-g}\left(R - 2(\nabla\phi)^2 - e^{-2\phi}F_{\mu\nu}F^{\mu\nu}\right) \tag{3.1}$$

When the dilaton is present, the Reissner-Nordström metric is no longer a solution to the equations of motion. Black holes charged under a U(1) gauge field must also carry dilatonic charge, as was analyzed by Garfinkle et al. (GHS) [15]. A magnetically charged black hole ($F = Q\sin\theta d\theta \wedge d\varphi$) is then described by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r\left(r - \frac{Q^2\,e^{-2\phi_0}}{M}\right)d\Omega^2 \tag{3.2}$$

$$e^{-2\phi} = e^{-2\phi_0}\left(1 - \frac{Q^2\,e^{-2\phi_0}}{Mr}\right) \tag{3.3}$$

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where ϕ_0 is the asymptotic value of ϕ at infinity, which we set to zero, for simplicity. The black hole has a horizon at $r = 2M$ for $M > |Q|/\sqrt{2}$. The solution for the dilaton implies that the black hole has a dilatonic charge of $D = -Q^2/2M$, which for the extremal case reduces to $D = -M$. The force between two particles with magnetic charge Q, dilatonic charge D, and mass M, is given by

$$F = \frac{Q^2 - D^2 - M^2}{16\pi r^2} \tag{3.4}$$

so the net force between two extremal black holes with equal charges vanishes. The argument of Arkani-Hamed et al. would then predict that higher-order corrections to the mass and the dilatonic charge would make the mass smaller and the net force repulsive as the charge Q becomes smaller.

Corrections to the metric and dilaton field of a magnetically-charged GHS black hole due to the next order terms (R^2, F^4, $F^2(\nabla\phi)^2$) in the heterotic string effective action have been calculated by Natsuume [16]. After eliminating many of the terms by field redefinitions, he obtained the corrections to leading order in α' as

$$\mathcal{L} = a\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) + b(F^2)^2 + cF^2(\nabla\phi)^2 + hR^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{3.5}$$

The coefficients of $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $R^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, which are invariant under field redefinitions, were then taken from the heterotic string calculations [13]: $a = \alpha'/8$ and $h = 0$. The perturbed equations of motion were written down, and a requirement of consistency with exact results that were obtained for this black hole [17] determined $c = \alpha'/2$. The value of b does not affect the correction to the mass. The metric (in the extremal limit) becomes

$$ds^2 = -\left(1 - \frac{\widetilde{Q}}{r}\right)^{1+\epsilon} f_2\left(\frac{\widetilde{Q}}{r}\right) dt^2 + \left(1 - \frac{\widetilde{Q}}{r}\right)^{-1+\epsilon} f_3\left(\frac{\widetilde{Q}}{r}\right) dr^2 + \tag{3.6}$$
$$+ r^2\left(1 - \frac{\widetilde{Q}}{r}\right)^{1+\epsilon} f_4\left(\frac{\widetilde{Q}}{r}\right) d\Omega^2$$

and the dilaton is given by

$$e^{-2\phi} = \left(1 - \frac{\widetilde{Q}}{r}\right)^{1+\epsilon} f_4\left(\frac{\widetilde{Q}}{r}\right) \tag{3.7}$$

where $\widetilde{Q} = \sqrt{2}\,Q$, $\epsilon = (2b-1)\alpha'/\widetilde{Q}^2$, and

$$f_2(x) = 1 - \frac{\alpha'}{40\widetilde{Q}^2}x(11x^3 + 7x^2 + 16x + 38) + g(x) \tag{3.8}$$

$$f_3(x) = 1 - \frac{\alpha'}{40\widetilde{Q}^2}x(19x^3 + 25x^2 + 26x + 42) + g(x) \tag{3.9}$$

$$f_4(x) = 1 - \frac{\alpha'}{40\widetilde{Q}^2}x(-9x^3 + 7x^2 + 16x + 38) + g(x) \tag{3.10}$$

$$g(x) = \frac{\alpha'}{60\widetilde{Q}^2}bx(15x^3 + 32x^2 + 57x + 120). \tag{3.11}$$

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Natsuume found that the mass-charge relation for the extremal black holes (with the normalization given in our eq. (3.1)) is given by

$$M = \frac{|Q|}{\sqrt{2}}\left(1 - \frac{\alpha'}{40Q^2}\right). \tag{3.12}$$

This agrees with the expectation that the M/Q ratio decreases as the charge Q becomes smaller.

Furthermore, we can use eq. (3.7) to determine the correction to the dilatonic charge D. We identify D as the coefficient of the $1/r^2$ term in $d\phi/dr$ and obtain the corrected dilatonic charge of the extremal black hole as

$$D = -\frac{|Q|}{\sqrt{2}}\left(1 - \frac{\alpha'}{40Q^2}\right). \tag{3.13}$$

Since both the mass and the dilatonic charge decrease, the net force (3.4) between the extremal black holes becomes repulsive, as was conjectured in section 1.

4. Discussion

We have calculated the corrections to the masses of extremal black holes in several backgrounds. In all examples where we could verify the sign, the sign was negative. This fact was not guaranteed by the general rules of effective field theory; however, general arguments exist why such an inequality could follow from the consistency of couplings in quantum gravity [2].

Other examples of black objects where the inequality could be checked include non-supersymmetric black holes in type II string theory on Calabi-Yau manifolds and various black branes. It is desirable to find either a more general proof that the extremal black holes become lighter in general backgrounds of quantum gravity or a counterexample. We also conjecture that the first correction to the Bekenstein-Hawking entropy, arising from higher-derivative terms applied to Wald's formula, is positive in all cases. We are not aware of counterexamples; explicit checks or a more general proof could shed some light on the UV-IR relations in quantum gravity.

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A. Energy-momentum tensor

First order corrections to the energy-momentum tensor $T_{\mu\nu}$ have two contributions: a correction to the energy-momentum tensor of the $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ term due to corrections to

$F^{\mu\nu}$, and an effective contribution representing the modification of Einstein's equation by all the higher-order terms.

To find the first contribution, we vary the action with respect to A_μ to obtain the corrected Maxwell's equations:

$$\begin{aligned}\nabla_\nu F^{\mu\nu} &= 4c_4\nabla_\nu(RF^{\mu\nu}) + 2c_5\nabla_\nu(R^{\mu\rho}F_\rho{}^\nu - R^{\nu\rho}F_\rho{}^\mu) \\ &\quad +4c_6\nabla_\nu(R^{\alpha\beta\mu\nu}F_{\alpha\beta}) + 8c_7\nabla_\nu(F_{\rho\sigma}F^{\rho\sigma}F^{\mu\nu}) \\ &\quad -4c_8\nabla_\nu\Box F^{\mu\nu} - 2c_9\nabla_\nu\nabla_\rho(\nabla^\mu F^{\rho\nu} - \nabla^\nu F^{\rho\mu})\end{aligned} \tag{A.1}$$

We find the first-order correction to $F^{\mu\nu}$ by treating the right hand side as a perturbation (evaluated with the unperturbed metric and electric field). Since $T_{\mu\nu}$ is quadratic in the fields, only corrections to F^{01} (which is non-zero in the unperturbed solution) are of the first order in c_i.

To find the second contribution to $T_{\mu\nu}$, we calculate the variation of the higher-order terms in the action with respect to $g^{\mu\nu}$, which gives

$$\begin{aligned}\Delta T_{\mu\nu} &= c_1\left(g_{\mu\nu}R^2 - 4RR_{\mu\nu} + 4\nabla_\nu\nabla_\mu R - 4g_{\mu\nu}\Box R\right) \\ &\quad +c_2\left(g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} + 4\nabla_\alpha\nabla_\nu R^\alpha_\mu - 2\Box R_{\mu\nu} - g_{\mu\nu}\Box R - 4R^\alpha_\mu R_{\alpha\nu}\right) \\ &\quad +c_3\Big(g_{\mu\nu}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} - 8\Box R_{\mu\nu} + 4\nabla_\nu\nabla_\mu R + 8R^\alpha_\mu R_{\alpha\nu} - 8R^{\alpha\beta}R_{\mu\alpha\nu\beta}\Big) \\ &\quad +c_4\left(g_{\mu\nu}RF^2 - 4RF_\mu{}^\sigma F_{\nu\sigma} - 2F^2R_{\mu\nu} + 2\nabla_\mu\nabla_\nu F^2 - 2g_{\mu\nu}\Box F^2\right) \\ &\quad +c_5\Big(g_{\mu\nu}R^{\kappa\lambda}F_{\kappa\rho}F_\lambda{}^\rho - 4R_{\nu\sigma}F_{\mu\rho}F^{\sigma\rho} - 2R^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - g_{\mu\nu}\nabla_\alpha\nabla_\beta(F^\alpha{}_\rho F^{\beta\rho}) \\ &\qquad + 2\nabla_\alpha\nabla_\nu(F_{\mu\beta}F^{\alpha\beta}) - \Box(F_{\mu\rho}F_\nu{}^\rho)\Big) \\ &\quad +c_6\Big(g_{\mu\nu}R^{\kappa\lambda\rho\sigma}F_{\kappa\lambda}F_{\rho\sigma} - 6F_{\alpha\nu}F^{\beta\gamma}R^\alpha{}_{\mu\beta\gamma} - 4\nabla_\beta\nabla_\alpha(F^\alpha{}_\mu F^\beta{}_\nu)\Big) + \\ &\quad +c_7\left(g_{\mu\nu}(F^2)^2 - 8F^2F_\mu{}^\sigma F_{\nu\sigma}\right) \\ &\quad +c_8\Big(g_{\mu\nu}(\nabla_\kappa F_{\rho\sigma})(\nabla^\kappa F^{\rho\sigma}) - 2(\nabla_\mu F_{\alpha\beta})(\nabla_\nu F^{\alpha\beta}) - 4(\nabla_\alpha F_{\beta\mu})(\nabla^\alpha F^\beta{}_\nu) \\ &\qquad + 4\nabla_\alpha(F_{\nu\beta}\nabla^\alpha F_\mu{}^\beta) + 4\nabla_\alpha(F_{\nu\beta}\nabla_\mu F^{\alpha\beta}) - 4\nabla_\alpha(F^\alpha{}_\beta\nabla_\nu F_\mu{}^\beta)\Big) \\ &\quad +c_9\Big(g_{\mu\nu}(\nabla_\kappa F_{\rho\sigma})(\nabla^\rho F^{\kappa\sigma}) - 4(\nabla_\mu F^{\alpha\beta})(\nabla_\alpha F_{\nu\beta}) - 2(\nabla_\alpha F_{\beta\mu})(\nabla^\beta F^\alpha{}_\nu) \\ &\qquad + 2\nabla_\alpha(F_{\nu\beta}\nabla^\alpha F_\mu{}^\beta) + 2\nabla_\alpha(F_{\nu\beta}\nabla_\mu F^{\alpha\beta}) - 2\nabla_\alpha(F^\alpha{}_\beta\nabla_\nu F_\mu{}^\beta)\Big)\end{aligned} \tag{A.2}$$

where we denoted $F^2 \equiv F_{\rho\sigma}F^{\rho\sigma}$.

B. Corrections to the Reissner-Nordström black hole in D dimensions

The solution presented in section 2 can be easily generalized to Reissner-Nordstöm black holes in D spacetime dimensions. (The unperturbed solution is presented in refs. [18] and [19].) The most general spherically symmetric static metric in D spacetime dimensions has the form

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2_{(D-2)} \tag{B.1}$$

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where

$$d\Omega^2_{(1)} = d\theta_0^2, \qquad d\Omega^2_{(i+1)} = d\theta_i^2 + \sin^2\theta_i \, d\Omega^2_{(i)} \qquad 0 \le \theta_0 \le 2\pi, \quad 0 \le \theta_i \le \pi$$

so the metric for the coordinates $(t,\, r,\, \theta_{D-3},\, \theta_{D-2},\, \ldots,\, \theta_0)$ is

$$g_{\mu\nu} = \text{diag}\left(-e^{\nu(r)},\, e^{\lambda(r)},\, r^2,\, r^2\sin^2\theta_{D-3},\, \ldots,\, r^2\sin^2\theta_{D-3}\cdots\sin^2\theta_2\sin^2\theta_1\right) \tag{B.2}$$

$$\sqrt{-g} = r^{D-2}\, e^{(\nu+\lambda)/2} \prod_{i=1}^{D-3} (\sin\theta_i)^i \tag{B.3}$$

The corresponding Christoffel symbols are

$$\Gamma^0_{00} = \Gamma^0_{0k} = \Gamma^0_{11} = \Gamma^0_{1k} = \Gamma^0_{kk'} = 0 \qquad \Gamma^0_{01} = \frac{\nu'}{2} \tag{B.4}$$

$$\Gamma^1_{00} = \frac{1}{2}\nu' e^{\nu-\lambda} \qquad \Gamma^1_{01} = \Gamma^1_{0k} = \Gamma^1_{1k} = \Gamma^1_{kk'|k'\neq k} = 0 \qquad \Gamma^1_{11} = \frac{\lambda'}{2} \qquad \Gamma^1_{kk} = -\frac{e^{-\lambda}}{r} g_{kk}$$

$$\Gamma^k_{00} = \Gamma^k_{01} = \Gamma^k_{0k'} = \Gamma^k_{11} = \Gamma^k_{1k'|k'\neq k} = 0 \qquad \Gamma^k_{1k} = \frac{1}{r} \qquad \Gamma^k_{\text{else}} \ \text{not shown}$$

where $k, k' = 2, \ldots, D-1$. The non-zero components of the Ricci tensor are

$$\begin{aligned}
R^0_0 &= -\frac{e^{-\lambda}}{2}\left(\nu'' + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2} + (D-2)\frac{\nu'}{r}\right) \\
R^1_1 &= -\frac{e^{-\lambda}}{2}\left(\nu'' + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2} - (D-2)\frac{\lambda'}{r}\right) \\
R^k_k &= -e^{-\lambda}\left(\frac{(D-3)(1-e^{\lambda})}{r^2} + \frac{\nu'-\lambda'}{2r}\right)
\end{aligned} \tag{B.5}$$

$$R = -e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2} + (D-2)(D-3)\frac{1-e^{\lambda}}{r^2} + (D-2)\frac{\nu'-\lambda'}{r}\right) \tag{B.6}$$

We can then write

$$\begin{aligned}
\frac{R^0_0 - R^1_1}{D-2} - R^k_k &= \frac{D-3}{r^2}(e^{-\lambda}-1) - \frac{\lambda' e^{-\lambda}}{r} = \frac{\left(r^{D-3}(e^{-\lambda}-1)\right)'}{r^{D-2}} \\
r^{D-3}(e^{-\lambda}-1) &= \int dr\, r^{D-2}\left(\frac{R^0_0 - R^1_1}{D-2} - R^k_k\right)
\end{aligned}$$

Assuming that the asymptotic behavior at $r \to \infty$ is the Schwarzschild solution, this becomes

$$e^{-\lambda} = 1 - \frac{2\kappa^2 M}{(D-2)\Omega_{D-2}\, r^{D-3}} - \frac{1}{r^{D-3}}\int_r^\infty dr\, r^{D-2}\left(\frac{R^0_0 - R^1_1}{D-2} - R^k_k\right) \tag{B.7}$$

where

$$\Omega_{D-2} = \frac{2\,\pi^{(D-1)/2}}{\Gamma[(D-1)/2]}$$

is the area of the unit sphere. Similarly,

$$\begin{aligned} R^0_0 - R^1_1 &= -(D-2)\frac{e^{-\lambda}}{2}\frac{\nu' + \lambda'}{r} \\ \nu &= -\lambda + \frac{2}{D-2}\int_r^\infty dr\, r\left(R^0_0 - R^1_1\right) e^{\lambda} \end{aligned} \tag{B.8}$$

Einstein's equation obtained from the action

$$S = \int d^D x\sqrt{-g}\,\frac{R}{2\kappa^2} + S_{\text{matter}} \tag{B.9}$$

can be written as

$$R_{\mu\nu} = \kappa^2\left(T_{\mu\nu} - \frac{T}{D-2}\, g_{\mu\nu}\right) \qquad\qquad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \tag{B.10}$$

and in our case $T = T^0_0 + T^1_1 + (D-2)T^k_k$. Then (B.7) and (B.8) become

$$e^{-\lambda} = 1 - \frac{2\kappa^2 M}{(D-2)\Omega_{D-2}\, r^{D-3}} - \frac{2\kappa^2}{(D-2)\, r^{D-3}}\int_r^\infty dr\, r^{D-2}\, T^0_0 \tag{B.11}$$

$$\nu = -\lambda + \frac{2\kappa^2}{D-2}\int_r^\infty dr\, r(T^0_0 - T^1_1)\, e^{\lambda} \tag{B.12}$$

The unperturbed electrically charged solution is

$$\begin{aligned} e^{\nu} &= e^{-\lambda} = 1 - \frac{2}{(D-2)\,\Omega_{D-2}}\frac{\kappa^2 M}{r^{D-3}} + \frac{1}{(D-2)(D-3)\Omega^2_{D-2}}\frac{\kappa^2 Q^2}{r^{2(D-3)}} \\ E &= \frac{Q}{\Omega_{D-2}\, r^{D-2}} \end{aligned} \tag{B.13}$$

We now consider an action of the form

$$\begin{aligned} S = \int d^D x\sqrt{-g}\Big(&\frac{R}{2\kappa^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_1\, R^2 + c_2\, R_{\mu\nu}R^{\mu\nu} + c_3\, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \\ &+ c_4\, RF_{\mu\nu}F^{\mu\nu} + c_5\, R^{\mu\nu}F_{\mu\rho}F_{\nu}{}^{\rho} + c_6\, R^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + c_7\,(F_{\mu\nu}F^{\mu\nu})^2 \\ &+ c_8\,(\nabla_\mu F_{\rho\sigma})(\nabla^\mu F^{\rho\sigma}) + c_9\,(\nabla_\mu F_{\rho\sigma})(\nabla^\rho F^{\mu\sigma})\Big) \end{aligned} \tag{B.14}$$

By the same procedure as described in the main text, we consider the corrections to the effective $T_{\mu\nu}$ based on the equations in appendix A, and obtain the mass-charge relation for extremal black holes

$$\begin{aligned} \frac{D-3}{D-2}\frac{\kappa^2 M^2}{Q^2} = 1 - &\frac{2(D-3)}{(D-2)(3D-7)}\left(\frac{(D-2)(D-3)\Omega^2_{D-2}}{\kappa^2 Q^2}\right)^{1/(D-3)} \times \\ &\times \big[(D-3)(D-4)^2\,\kappa^2 c_1 + (D-3)(2D^2-11D+16)\,\kappa^2 c_2 + \\ &+ 2(2D^3-16D^2+45D-44)\,\kappa^2 c_3 + 2(D-2)(D-3)(D-4)c_4 + \\ &+ 2(D-2)(D-3)^2 c_5 + 2(D-2)(D-3)^2 c_6 + 4(D-2)^2(D-3)\frac{c_7}{\kappa^2} - \\ &- 2(D-2)(D-3)^2 c_8 - (D-2)(D-3)^2 c_9\big] \end{aligned} \tag{B.15}$$

It is convenient to choose the normalization $\kappa^2 = (D-2)/(D-3)$, and then

$$\frac{M^2}{Q^2} = 1 - \frac{2}{3D-7}\left(\frac{(D-3)\,\Omega_{D-2}}{Q}\right)^{2/(D-3)} \times \tag{B.16}$$
$$\times\left[(D-3)(D-4)^2 c_1 + (D-3)(2D^2-11D+16)c_2 + \right.$$
$$+2(2D^3-16D^2+45D-44)c_3 + 2(D-3)^2(D-4)c_4 + 2(D-3)^3 c_5 +$$
$$\left. +2(D-3)^3 c_6 + 4(D-3)^3 c_7 - 2(D-3)^3 c_8 - (D-3)^3 c_9\right]$$

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