

The Asymptotic Weak Gravity Conjecture in M-theory

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based mainly on [arXiv:2212.09758](https://arxiv.org/abs/2212.09758) [hep-th]

with C. F. Cota, T. Weigand and M. Wiesner

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Weak Gravity Conjecture

Weak Gravity Conjecture

Every gauge theory coupled to gravity has a particle with $\frac{g_{\text{YM}}^2 q^2}{m^2} \geq \frac{g_{\text{YM}} Q^2}{M^2} \Big|_{\text{B.H.}}$.¹

Repulsive Force Conjecture

States with highest charge-to-mass ratio should be self-repulsive:² $F_{\text{Coulomb}} \geq F_{\text{Grav.}} + F_{\text{Yukawa}}$

Tower Weak Gravity Conjecture

The super-extremal states must form an infinite tower.³

¹N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: [hep-th/0601001](#) (hep-th)

²N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: [hep-th/0601001](#) (hep-th); E. Palti, *JHEP* **08**, 034, arXiv: [1705.04328](#) (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: [1906.02206](#) (hep-th)

³B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **02**, 140, arXiv: [1509.06374](#) (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *Phys. Rev. Lett.* **121**, 051601, arXiv: [1802.08698](#) (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: [1906.02206](#) (hep-th); M. Montero, G. Shiu, P. Soler, *JHEP* **10**, 159, arXiv: [1606.08438](#) (hep-th); S. Andriolo, D. Junghans, T. Noumi, G. Shiu, *Fortsch. Phys.* **66**, 1800020, arXiv: [1802.04287](#) (hep-th)

Evidence for Tower WGC

To verify the Tower WGC, one must compute the charge-to-mass ratio of candidates of super-extremal states.

This can be achieved when the charge-to-mass ratio is known:

- ① Towers of BPS particles in 4d $\mathcal{N} = 2,^4$ or 5d $\mathcal{N} = 1$ theories.⁵
- ② Perturbative heterotic string.⁶
- ③ F-theory in the limits where $g_{\text{YM}} \rightarrow 0$.⁷

Timo's talk!

⁴T. W. Grimm, E. Palti, I. Valenzuela, *JHEP* **08**, 143, arXiv: 1802.08264 (hep-th); N. Gendler, I. Valenzuela, *JHEP* **01**, 176, arXiv: 2004.10768 (hep-th); B. Bastian, T. W. Grimm, D. van de Heisteeg, *JHEP* **06**, 162, arXiv: 2011.08854 (hep-th).

⁵M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th).

⁶N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: hep-th/0601001 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **02**, 140, arXiv: 1509.06374 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *Phys. Rev. Lett.* **121**, 051601, arXiv: 1802.08698 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: 1906.02206 (hep-th); M. Montero, G. Shiu, P. Soler, *JHEP* **10**, 159, arXiv: 1606.08438 (hep-th); S. Andriolo, D. Junghans, T. Noumi, G. Shiu, *Fortsch. Phys.* **66**, 1800020, arXiv: 1802.04287 (hep-th).

⁷S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **10**, 164, arXiv: 1808.05958 (hep-th); S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **08**, 104, arXiv: 1901.08065 (hep-th); S.-J. Lee, W. Lerche, T. Weigand, *Nucl. Phys. B* **938**, 321–350, arXiv: 1810.05169 (hep-th).

The Asymptotic WGC

The charge-to-mass ratio of non-BPS states is calculated **reliably** only near $g_{\text{YM}} \rightarrow 0$, and we referred to such WGC as **asymptotic** WGC.

The asymptotic WGC with minimal supersymmetry, has been studied in F-theory compactification to 6d⁸ and 4d theories⁹ where

- ① No BPS particle states are available, but **non-BPS excitations** of a weakly coupled BPS string are.
- ② In 4d, super-extremal states exists in emergent string limits where a BPS string dual to a heterotic string becomes light.
- ③ In particular, there are cases in 4d, where no particle-like excitations of BPS strings can furnish a tower of super-extremal tower of states.
- ④ This happens when

$$\frac{\Lambda_{\text{WGC}}^2}{\Lambda_{\text{sp}}^2} \equiv \frac{g_{\text{YM}}^2 M_{\text{Pl}}^{d-2}}{\Lambda_{\text{sp}}^2} \Big|_{d=4}$$

does not vanish asymptotically.

Timo's talk!

⁸S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **10**, 164, arXiv: 1808.05958 (hep-th); S.-J. Lee, W. Lerche, T. Weigand, *Nucl. Phys. B* **938**, 321–350, arXiv: 1810.05169 (hep-th).

⁹S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **08**, 104, arXiv: 1901.08065 (hep-th); D. Klaeuer, S.-J. Lee, T. Weigand, M. Wiesner, *JHEP* **03**, 252, arXiv: 2011.00024 (hep-th); C. F. Cota, A. Mininno, T. Weigand, M. Wiesner, *JHEP* **11**, 058, arXiv: 2208.00009 (hep-th).

Goal of the Presentation

Puzzle for tower WGC in 5d

There are directions in the charge lattice where there is **no** BPS tower.

- 1 We extend the discussion of the asymptotic WGC to 5d $\mathcal{N} = 1$ theories, given by M-theory on Calabi–Yau 3-fold.
- 2 We focus on the role of super-extremal **non-BPS towers**, complementing the potential lack of super-extremal BPS states¹⁰ in **all** directions in the charge lattice dual to the gauge groups with a weak coupling limit.
- 3 Our main interest lies on those directions in the charge lattice that are **not** populated by towers of BPS states, with the goal of showing explicitly that the asymptotic Tower WGC holds along such non-BPS directions.

¹⁰M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th).

Summary of Results

- 1 We **refine** the classification of infinite distance limits¹¹ for tests of the asymptotic WGC.
- 2 We **classify** the linear combinations of U(1) factors that become weakly coupled in the respective limit.

Upshots

- 3 U(1)s can **only** become weakly coupled in a given infinite distance limit if they are obtained from reducing C_3 over
 - a curve that is contained in the **generic fiber** or
 - a curve that is localized in a degenerate fiber associated to a **finite** distance degeneration in the deformation space of the generic fiber.
- 4 **Any direction** in the charge lattice of five-dimensional M-theory that is **not** populated by a tower of BPS states either carries charge under a U(1) **without** a weak coupling limit, or there are super-extremal non-BPS states arising as excitations of a **critical string**.

¹¹S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **02**, 190, arXiv: 1910.01135 (hep-th).

Contents

- 1 Weak Coupling Limits
- 2 (Non-)BPS Tower and Weak Gravity Conjecture
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M-theory on Calabi–Yau 3-folds

$$S_{5d} = \frac{M_{\text{Pl}}^3}{2} \int_{\mathbb{R}^{1,4}} \left(R \star \mathbb{1} - g_{AB} d\Phi^A \wedge \star d\Phi^B \right) - \frac{1}{2g_5^2} \int_{\mathbb{R}^{1,4}} f_{\alpha\beta} F^\alpha \wedge \star F^\beta + \dots,$$

with

$$M_{\text{Pl}}^3 = 4\pi M_{11d}^3 \mathcal{V}, \quad \frac{1}{g_5^2} = \frac{M_{\text{Pl}}}{(2\pi)(4\pi)^{1/3}}, \quad f_{\alpha\beta} = \frac{1}{\mathcal{V}^{1/3}} \left(\frac{\mathcal{V}_\alpha \mathcal{V}_\beta}{\mathcal{V}} - \mathcal{V}_{\alpha\beta} \right) = \left(\widehat{\mathcal{V}}_\alpha \widehat{\mathcal{V}}_\beta - \widehat{\mathcal{V}}_{\alpha\beta} \right).$$

- ① Φ^A are $h^{1,1}(X_3) - 1$ scalar fields functions of the Kähler moduli v^a in

$$J = \sum_{\alpha=1}^{h^{1,1}(X_3)} v^\alpha J_\alpha.$$

- ② F^α are the field strengths of the U(1) gauge factors.
 ③ A generic U(1)_C for a curve class $C \in H_2(X_3)$ is

$$U(1)_C = c_\alpha U(1)^\alpha, \quad \text{with } g_{\text{YM},C}^2 = g_5^2 c_\alpha f^{\alpha\beta} c_\beta, \quad \text{for } C = c_\alpha C^\alpha \in H_2(X_3), \quad \text{with } J_\alpha \cdot C^\beta = \delta_\alpha^\beta.$$

Repulsive Force Conjecture

In the weak coupling limit, for a particle of mass M_k , we require¹²

Repulsive Force Conjecture

$$\frac{(M_{\text{Pl}} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2} \geq \frac{1}{M_{\text{Pl}}^2} \frac{d-3}{d-2} \Big|_{d=5} + \frac{1}{4} \frac{M_{\text{Pl}}^2}{M_k^4} \left(f^{\alpha\beta} - \frac{1}{3} \hat{v}^\alpha \hat{v}^\beta \right) \partial_\alpha \left(\frac{M_k^2}{M_{\text{Pl}}^2} \right) \partial_\beta \left(\frac{M_k^2}{M_{\text{Pl}}^2} \right)$$

¹²N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: [hep-th/0601001](https://arxiv.org/abs/hep-th/0601001) (hep-th); E. Palti, *JHEP* **08**, 034, arXiv: [1705.04328](https://arxiv.org/abs/1705.04328) (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: [1906.02206](https://arxiv.org/abs/1906.02206) (hep-th).

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In the weak coupling limit, for a particle of mass M_k , we require¹²

Repulsive Force Conjecture

$$\underbrace{\frac{(M_{\text{Pl}} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2}}_{F_{\text{Coulomb}}} \geq \underbrace{\frac{1}{M_{\text{Pl}}^2} \frac{d-3}{d-2}}_{F_{\text{Gravitational}}}\Big|_{d=5} + \underbrace{\frac{1}{4} \frac{M_{\text{Pl}}^2}{M_k^4} \left(f^{\alpha\beta} - \frac{1}{3} \hat{v}^\alpha \hat{v}^\beta \right)}_{F_{\text{Yukawa}}} \partial_\alpha \left(\frac{M_k^2}{M_{\text{Pl}}^2} \right) \partial_\beta \left(\frac{M_k^2}{M_{\text{Pl}}^2} \right)$$

¹²N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: hep-th/0601001 (hep-th); E. Palti, *JHEP* **08**, 034, arXiv: 1705.04328 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: 1906.02206 (hep-th).

Weak Coupling Limits

- 1 The magnetic WGC associates a scale Λ_{WGC} to a gauge theory in d dimensions with (dimensionful) gauge coupling g_{YM} , which is given by

$$\Lambda_{\text{WGC}}^2 = g_{\text{YM}}^2 M_{\text{Pl}}^{d-2}.$$

- 2 For any $U(1)_C$, then

$$\Lambda_{\text{WGC}}^2(U(1)_C) = g_{\text{YM},C}^2 M_{\text{Pl}}^3 = g_5^2 (c_\alpha f^{\alpha\beta} c_\beta) M_{\text{Pl}}^3,$$

- 3 and the **weak coupling limit** for the gauge group $U(1)_C$ now corresponds to the limit

$$\frac{\Lambda_{\text{WGC}}^2(U(1)_C)}{\Lambda_{\text{QG}}^2} \rightarrow 0,$$

where Λ_{QG} is the quantum gravity cut-off, i.e., the scale at which gravity becomes strongly coupled.

Dieter and Timo's talks!

Main Claim

Main Claim

Consider M-theory compactified to five dimensions on a Calabi–Yau X_3 . Suppose there exists a primitive charge vector $\mathbf{Q}^0 \in \Lambda_{\mathbf{Q}}$ such that $\{\lambda \mathbf{Q}^0\}_{\lambda \in \mathbb{R}} \cap \Lambda_{\mathbf{Q}}$ is **not** populated by a BPS tower of super-extremal states. Defining $U(1)_{\mathbf{Q}^0} = Q_a^0 U(1)^a$ one of the following two holds:

- 1 There exists **no** limit in moduli space in which

$$\frac{\Lambda_{\text{WGC}}^2(U(1)_{\mathbf{Q}^0})}{\Lambda_{\text{QG}}^2} \rightarrow 0,$$

with $\Lambda_{\text{WGC}}^2(U(1)_C) = g_{\text{YM,C}}^2 M_{\text{Pl}}^3 = g_5^2 (c_\alpha f^{\alpha\beta} c_\beta) M_{\text{Pl}}^3$.

- 2 Alternatively, there does exist a **non-BPS** tower of states along the ray in the charge lattice which is part of the tower of excitations of a **critical string** becoming weakly coupled in the limit, and this tower of states is self-repulsive in the asymptotic limit in the sense of **satisfying condition** of the RFC.

Understanding the Claim

Whenever there **exists** a limit of the type

$$\frac{\Lambda_{\text{WGC}}^2(U(1)_{\mathbf{Q}^0})}{\Lambda_{\text{QG}}^2}$$

either

- there **exists** a BPS tower of super-extremal states along the given ray in the charge lattice, or
- we can identify a **non-BPS** tower of self-repulsive states originating in the excitation spectrum of an asymptotically weakly coupled **critical string**.

- 1 Since $\Lambda_{\text{QG}} \leq M_{\text{Pl}}$, then

$$Q_{\alpha}^0 f^{\alpha\beta} Q_{\beta}^0 \rightarrow 0.$$

- 2 We look for **all possible limits** in the 5d moduli space and compare with the scaling of Λ_{QG} .
- 3 Rescaling of all the Kähler moduli

$$v^{\alpha} \rightarrow \lambda \tilde{v}^{\alpha}, \quad \forall \alpha = 1, \dots, h^{1,1}(X_3),$$

we can assume that the overall volume \mathcal{V} stays constant along the trajectory in moduli space.

- 4 Hence it suffices to analyze limits in the **vector moduli space**.

Type of Limits

Limits in vector moduli space

- ① **Limits of Type T^2 :**¹³ X_3 admits a torus fibration

$$\pi : X_3 \rightarrow B_2$$

and the weak-coupling limit corresponds to a limit in which the volume of the generic fiber T^2 shrinks as

$$\mathcal{V}_{T^2} \sim \frac{1}{\lambda}, \quad \mathcal{V}_{B_2} \sim \lambda, \quad \lambda \rightarrow \infty.$$

- ② **Limits of Type $K3/T^4$:**¹³ X_3 allows for a surface fibration

$$\rho : X_3 \rightarrow \mathbb{P}^1$$

with generic fiber \mathbf{S} being either a K3 surface, or an abelian surface, T^4 . In the weak coupling limit, the volume of the surface fiber shrinks as

$$\mathcal{V}_{K3/T^4} \sim \frac{1}{\lambda^2}, \quad \mathcal{V}_{\mathbb{P}^1} \sim \lambda^2, \quad \lambda \rightarrow \infty.$$

¹³S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **02**, 190, arXiv: 1910.01135 (hep-th)

Results

T^2 -type Limits

Weakly coupled U(1)s: U(1) associated to the generic T^2 -fiber, i.e. KK U(1) of the lift to F-theory on the base of the CY 3-fold times S^1 . These are **BPS states**, reflecting the fact that the generic T^2 fiber is a movable curve.¹⁴

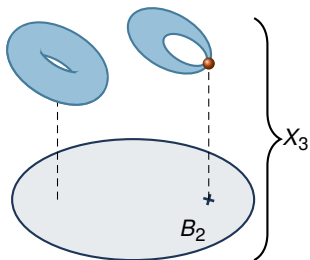
$K3/T^4$ -type Limits

Weakly coupled U(1)s: U(1)s obtained by reducing C_3 over a curve in a generic $K3/T^4$ fiber of X_3 or curves contained in a degenerate fiber arising at **finite** distance in the fiber moduli space. The charged states can all be interpreted as **excitations of the heterotic string** obtained by wrapping an M5-brane on the generic fiber.

¹⁴M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th)

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T^2 -type LimitsLimits of Type T^2

X_3 admits a torus fibration

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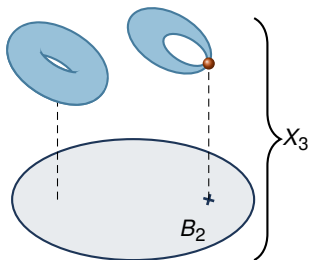
and the weak-coupling limit corresponds to a limit in which the volume of the generic fiber T^2 shrinks as

$$\mathcal{V}_{T^2} \sim \frac{1}{\lambda}, \quad \mathcal{V}_{B_2} \sim \lambda, \quad \lambda \rightarrow \infty.$$

- 1 Mori cone generators: $\{C^\alpha\} = \{C^a, C_f^i\}$, $a = 1, \dots, h^{1,1}(B_2)$, $i = 1, \dots, n$.
- 2 Basis of $H^{1,1}(X_3)$: $\{J_\alpha\} = \{J_a, J_i\}$ with $J_a = \pi^*(j_a)$.
- 3 Gauge kinetic matrix

$$f_{\alpha\beta} \sim \begin{pmatrix} \tilde{f}_{ab}\lambda^{-1} & \tilde{f}_{ai}\lambda^{1/2} \\ \tilde{f}_{ia}\lambda^{1/2} & \tilde{f}_{ij}\lambda^2 \end{pmatrix}$$

T^2 -type Limits - BPS Tower



- 1 The **only** asymptotically weakly coupled linear combination is

$$U(1)_\mathcal{E} = \sum_{i=1}^n c_i U(1)^i \text{ with } \frac{\Lambda_{\text{WGC}}^2(U(1)_\mathcal{E})}{\Lambda_{\text{sp, KK}}^2} \sim \frac{1}{\lambda^{3/2}}.$$

- 2 This $U(1)_\mathcal{E}$ clearly satisfies the asymptotic tower WGC conjecture because the ray in the charge lattice along $n\mathbf{Q}_\mathcal{E}$ is populated by a tower of **BPS particles**.¹⁵

- 3 These are the BPS states obtained by wrapping n M2-branes along the generic fiber, which furnish the KK tower for the effective five-dimensional to six-dimensional limit.
- 4 The BPS invariants, more precisely the genus-zero Gopakumar–Vafa invariants, for the curve $n\mathcal{E}$ coincide with the Euler characteristic of X_3 , i.e.,¹⁶

$$N_{n\mathcal{E}}^0 = -\chi(X_3).$$

¹⁵M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th)

¹⁶P.-K. Oehlmann, T. Schimannek, *JHEP* **09**, 066, arXiv: 1912.09493 (hep-th); A.-K. Kashani-Poor, *Commun. Math. Phys.* **386**, 1155–1207, arXiv: 1912.10009 (hep-th)

K3-type Limits

Limits of Type K3/T⁴

X_3 allows for a surface fibration

$$\rho : X_3 \rightarrow \mathbb{P}^1$$

with generic fiber \mathbf{S} being either a K3 surface, or an abelian surface, T^4 . In the weak coupling limit, the volume of the surface fiber shrinks as

$$\mathcal{V}_{\text{K3/T}^4} \sim \frac{1}{\lambda^2}, \quad \mathcal{V}_{\mathbb{P}^1} \sim \lambda^2, \quad \lambda \rightarrow \infty.$$

- 1 Mori cone generators: $\{C^\alpha\} = \{C^0, C^i, \quad i = 1, \dots, h^{1,1}(X_3) - 1\}$.
- 2 Basis of $H^{1,1}(X_3)$: $\{J_\alpha\} = \{J_0, J_i, \quad i = 1, \dots, h^{1,1}(X_3) - 1\}$.
- 3 Gauge kinetic matrix:

$$f_{\alpha\beta} \sim \begin{pmatrix} \tilde{f}_{00}\lambda^{-2} & \tilde{f}_{0i}\lambda^{-1/2} \\ \tilde{f}_{i0}\lambda^{-1/2} & \tilde{f}_{ij}\lambda \end{pmatrix}$$

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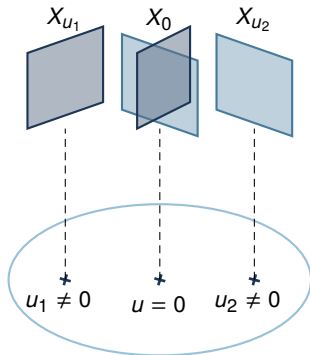
$$\mathcal{V}_{\text{K3/T}^4} \sim \frac{1}{\lambda^2}, \quad \mathcal{V}_{\mathbb{P}^1} \sim \lambda^2, \quad \lambda \rightarrow \infty.$$

- ① If $f_{ij} = \tilde{f}_{ij} \lambda = \lambda((\hat{v}^0)^2 \eta_{ik} \hat{v}^k \eta_{jl} \hat{v}^l - \hat{v}^0 \eta_{ij}) + \mathcal{O}(\lambda^{-1/2})$ has full rank, then for $U(1) = Q_i U(1)^i$

$$\frac{\Lambda_{\text{WGC}}^2(U(1))}{\Lambda_{\text{sp}}^2} \sim \frac{1/\lambda}{1/\lambda(1 + \log(\lambda))} \rightarrow 0 \quad \Lambda_{\text{sp}}^2 \sim M_{\text{het}}^2 \log\left(\frac{M_{\text{Pl}}}{M_{\text{het}}}\right)$$

- ② Tower given by excitations of the string given by M5-branes on K3-fiber (i.e. heterotic string), with charge $\mathbf{Q} = (0, Q_i)$.
- ③ If the fiber is not degenerate, or degenerate of Kulikov Type I, the intersection form η_{ij} (given by the intersection matrix on Λ_0) is **non-degenerate** $\implies \tilde{f}_{ij}$ has **full** rank and every fibral curve gives rise to weakly-coupled $U(1)$ s.

K3-type Limits - Degenerations of Type II/III



- Central fiber is reducible:

$$X_0 = \bigcup_{M=1}^N X_M.$$

- Mori cone generators localized in degenerate Type II/III fibers: C^μ .

- The K3-fiber degenerates at **infinite distance** over $p_{II/III} \in \mathbb{P}^1$.
- Reducible fiber implies divisors D_ρ in the degenerate fibers:

$$J_0 = \sum_{\rho} a_{\rho} D_{\rho}.$$

- Curves C splitting over $p_{II/III} \in \mathbb{P}^1$ are

$$C = \sum_{\mu} c_{\mu} C^{\mu} + C_{\text{rest}}$$

such that the dual divisors J_{μ} satisfies $c_{\nu} J_{\mu} = c_{\mu} J_{\nu} + \sum_{\rho} \alpha_{\rho} D_{\rho}$.

- However,

$$c_{\nu} J_{\mu} \cdot J_0 \cdot J_{\alpha} = c_{\mu} J_{\nu} \cdot J_0 \cdot J_{\alpha}.$$

- \tilde{f}_{ij} has **reduced** rank and the **only** weak coupling U(1) is given by

$$U(1)_C = \sum_{\mu} c_{\mu} U(1)^{\mu} + U(1)_{C_{\text{rest}}}, \quad \text{with } \Lambda_{\text{WGC}}^2(U(1)_C) \sim 1/\lambda.$$

K3-type Limits - Weakly Coupled U(1)s

Weakly coupled U(1)s obtained reducing C_3 over

- ① curves in the **generic K3-fiber**.
- ② curves in a degenerate fiber of Kulikov type I,¹⁷ i.e. at **finite distance** in the K3-fiber moduli space.
- ③ a curve C given by a **specific** linear combination of curves in a degenerate fiber of Kulikov type II or III.¹⁷

④ Generic K3-fiber or curves in irreducible degenerate fibers form a sublattice Λ^* of rank $(1, k_{0+I})$ of $\Gamma^{3,19}$, dual to the polarization lattice Λ of the K3-fibration $\rho : X_3 \rightarrow \mathbb{P}^1$, i.e. the charge lattice.

⑤ Let us denote

$$\Lambda_{\mathbb{R}}^* = \Lambda_+^* \oplus \Lambda_-^*,$$

such that $\mathbf{Q}_+ \in \Lambda_+^*$ satisfies $\mathbf{Q}_+^2 > 0$, while $\mathbf{Q}_-^2 < 0$ for $\mathbf{Q}_- \in \Lambda_-^*$.

¹⁷V. S. Kulikov, *Mathematics of the USSR-Izvestiya* 11, 957; V. S. Kulikov, *Mathematics of the USSR-Izvestiya* 17, 339; U. Persson, H. Pinkham, *Ann of Math. (2)* 113 01, 45–66

K3-type Limits - (Non-)BPS Towers

$$\Sigma \cdot_{K3} \Sigma \geq 0$$

- 1 Define rays in the charge lattice admitting towers of **BPS states**: M2-branes wrapped n times on Σ give rise to BPS particles¹⁸ with GV invariant $N_{n\Sigma}^0 \neq 0$ for $n \in \mathbb{N}$.
- 2 Such curves lie in the **movable cone** of X_3 because they are movable within the K3-fiber.¹⁹

$$\Sigma \cdot_{K3} \Sigma < 0$$

- 3 Define rays in the charge lattice which do **not** support a BPS tower.
- 4 Such curves are **rigid** within the K3-fiber and lie outside the movable cone of X_3 .¹⁹
- 5 **Existing** of a weak coupling limit, implies that the asymptotic Tower WGC calls for a super-extremal tower of states in these directions.
- 6 The non-BPS states are excitations of the **emergent heterotic string** which becomes light in the weak coupling limit.²⁰

¹⁸D. Maulik, R. Pandharipande, in *A celebration of algebraic geometry* (Amer. Math. Soc., Providence, RI, 2013), vol. 18, pp. 469–507

¹⁹M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th)

²⁰S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **02**, 190, arXiv: 1910.01135 (hep-th)

K3-type Limits - $\mathbf{Q} \in \Lambda_-^*$

- 1 The non-BPS states can be identified as excitations of the five-dimensional heterotic string obtained by wrapping an M5-brane on the K3-fiber in M-theory.
- 2 We identify the coefficients of the elliptic genus of the five-dimensional heterotic string at left-moving excitation level n and with charge vector $\mathbf{Q} \in \Lambda_-^*$ with certain D4-D2-D0 **Donaldson–Thomas** invariants in Type IIA string theory on X_3 .
- 3 The existence of these states then follows from the correspondence with **Noether–Lefschetz numbers**.²¹
- 4 There exists a distinguished set of non-BPS states at left-moving excitation level n_L and with (anti-self-dual) charge vector $\mathbf{Q} \in \Lambda_-^*$ obeying the relation

$$n_L = -\frac{1}{2} Q_i \eta^{ij} Q_j =: -\frac{1}{2} \mathbf{Q}^2,$$

where η^{ij} is the inverse of the intersection form on the K3-fiber.

²¹S. H. Katz, A. Klemm, C. Vafa, *Adv. Theor. Math. Phys.* **3**, 1445–1537, arXiv: [hep-th/9910181](https://arxiv.org/abs/hep-th/9910181); R. Pandharipande, R. P. Thomas, *Forum Math. Pi* **4**, e4, arXiv: [1404.6698](https://arxiv.org/abs/1404.6698) (math.AG); D. Maulik, R. Pandharipande, in *A celebration of algebraic geometry* (Amer. Math. Soc., Providence, RI, 2013), vol. 18, pp. 469–507; V. Bouchard, T. Creutzig, D.-E. Diaconescu, C. Doran, C. Quigley, A. Sheshmani, *Commun. Math. Phys.* **350**, 1069–1121, arXiv: [1601.04030](https://arxiv.org/abs/1601.04030) (hep-th).

K3-type Limits - $\mathbf{Q}_+ = 0$ and $\mathbf{Q}_- \neq 0$

- 1 We look for states with $n_L = -\frac{1}{2}\mathbf{Q}^2$.
- 2 We consider the elliptic genus of an MSW string²² obtained from M5-brane on K3-fiber:

$$Z_{\text{K3}}^{(r)}(\tau, \bar{\tau}, \mathbf{z}, \mathcal{B}) = \text{Tr}_{\text{RR}} F_{\text{R}}^2(-1)^{F_{\text{R}}} q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} e^{2\pi i z^i Q_i}.$$

- 3 We counted the BPS states obtained by wrapping the MSW string r times on an S^1 such that²³

$$Z_{\text{K3}}^{(r)}(\tau, \bar{\tau}, \mathbf{z}, \mathcal{B}) = \sum_{\mu \in \Lambda^*/r\Lambda} Z_{\mu}(\tau) \Theta_{\mu,r}^*(\tau, \bar{\tau}, \mathbf{z}, \mathcal{B}), \text{ where}$$

- $Z_{\mu}(\tau) = \sum_{n=0}^{\infty} \Omega(\gamma) q^{n + \mathbf{Q}^2/2r - 1}$ is a vector-modular form and $\Omega(\gamma)$ are **Donaldson-Thomas invariants**.
- $\Theta_{\mu,r}^*(\tau, \bar{\tau}, \mathbf{z}, \mathcal{B}) = \sum_{\lambda \in \mu + \sqrt{r}\Lambda} e^{-\pi i \tau (\lambda + \mathcal{B})_-^2 - \pi i \bar{\tau} (\lambda + \mathcal{B})_+^2 + 2\pi i \left(\lambda + \frac{\mathcal{B}}{2}\right) \cdot \mathbf{z}}$ is complex conjugate of Siegel theta series.

- 4 We use **Noether-Lefschetz theory**²⁴ to argue that for $n = -\frac{1}{2}\mathbf{Q}^2$, $\Omega(\gamma) \neq 0$.

²²J. M. Maldacena, A. Strominger, E. Witten, *JHEP* **12**, 002, arXiv: [hep-th/9711053](https://arxiv.org/abs/hep-th/9711053) (hep-th).

²³D. Gaiotto, A. Strominger, X. Yin, *JHEP* **08**, 070, arXiv: [hep-th/0607010](https://arxiv.org/abs/hep-th/0607010) (hep-th); V. Bouchard, T. Creutzig, D.-E. Diaconescu, C. Doran, C. Quigley, A. Sheshmani, *Commun. Math. Phys.* **350**, 1069–1121, arXiv: [1601.04030](https://arxiv.org/abs/1601.04030) (hep-th).

²⁴D. Maulik, R. Pandharipande, in *A celebration of algebraic geometry* (Amer. Math. Soc., Providence, RI, 2013), vol. 18, pp. 469–507.

K3-type Limits - Tower WGC

- 1 Consider the excitations of the heterotic string at **left-moving** excitation level n_L carrying charges Q_i within the charge lattice Λ^* .
- 2 The mass is given by

$$M_{n_L, \mathbf{Q}}^2 = 8\pi(n_L - a)T_s + \Delta_{\text{CB}} = \underbrace{16\pi^2(4\pi)^{-2/3}}_{\omega} \left((n_L - a)\widehat{\mathcal{V}}_s + \frac{1}{4}Q_i Q_j \widehat{v}^i \widehat{v}^j \right) M_{\text{Pl}}^2$$

- 3 Let us now analyze in more detail the special subset of states for which

$$n_L = -\frac{1}{2}Q_i \eta^{ij} Q_j$$

- 4 The RFC evaluates to

$$\text{RHS} \simeq 1 + \frac{\frac{1}{4n_L} Q_i Q_j \widehat{v}^i \widehat{v}^j}{\widehat{\mathcal{V}}_{\text{K3}} + \frac{1}{4n_L} Q_i Q_j \widehat{v}^i \widehat{v}^j} - \frac{\omega^2 n_L^2}{\widehat{v}^0} \frac{M_{\text{Pl}}^4}{M_{n_L, \mathbf{Q}}^4} \frac{1}{4n_L} Q_i Q_j \widehat{v}^i \widehat{v}^j \left(1 + \frac{1}{2n_L} (Q_k \eta^{kl} Q_l) \right)$$

$$\text{LHS} \simeq 1 + \frac{\frac{1}{4n_L} Q_i Q_j \widehat{v}^i \widehat{v}^j}{\widehat{\mathcal{V}}_{\text{K3}} + \frac{1}{4n_L} Q_i Q_j \widehat{v}^i \widehat{v}^j}$$

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$$\begin{aligned} \text{RHS} &\simeq 1 + \frac{\frac{1}{4n_L} Q_i Q_j \widehat{\mathcal{V}}^i \widehat{\mathcal{V}}^j}{\widehat{\mathcal{V}}_{K3} + \frac{1}{4n_L} Q_i Q_j \widehat{\mathcal{V}}^i \widehat{\mathcal{V}}^j} - \frac{\omega^2 n_L^2}{\widehat{\mathcal{V}}^0} \frac{M_{\text{Pl}}^4}{M_{n_L, \mathbf{Q}}^4} \frac{1}{4n_L} Q_i Q_j \widehat{\mathcal{V}}^i \widehat{\mathcal{V}}^j \left(1 + \frac{1}{2n_L} (Q_k \eta^{kl} Q_l) \right) \\ \text{LHS} &\simeq 1 + \frac{\frac{1}{4n_L} Q_i Q_j \widehat{\mathcal{V}}^i \widehat{\mathcal{V}}^j}{\widehat{\mathcal{V}}_{K3} + \frac{1}{4n_L} Q_i Q_j \widehat{\mathcal{V}}^i \widehat{\mathcal{V}}^j} \end{aligned}$$

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$$\text{RHS} \simeq 1 + \frac{\frac{1}{4n_L} Q_i Q_j \widehat{\mathcal{V}}^i \widehat{\mathcal{V}}^j}{\widehat{\mathcal{V}}_{\text{K3}} + \frac{1}{4n_L} Q_i Q_j \widehat{\mathcal{V}}^i \widehat{\mathcal{V}}^j}$$

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Contents

- 1 Weak Coupling Limits
- 2 (Non-)BPS Tower and Weak Gravity Conjecture
- 3 Conclusions**

Conclusions

- 1 We classified the linear combinations of $U(1)$ factors that become weakly coupled in the respective limit.
- 2 $U(1)$ s can **only** become weakly coupled in a given infinite distance limit if they are obtained from reducing C_3 over
 - a curve that is contained in the **generic fiber** or
 - a curve that is localized in a degenerate fiber associated to a **finite** distance degeneration in the deformation space of the generic fiber.
- 3 **Any direction** in the charge lattice of five-dimensional M-theory that is **not** populated by a tower of BPS states either carries charge under a $U(1)$ **without** a weak coupling limit, or there are super-extremal non-BPS states arising as excitations of a **critical string**.

Questions

- 4 What can we say about the tower WGC in the directions where there is tower of states?
- 5 When $\Lambda_{\text{WGC}} \geq \Lambda_{\text{sp}}$, is the tower required for the validity of the EFT? What kind of constraints can impose to the EFT?

Thank you!

$\mathbf{Q}_+ \neq 0$ and $\mathbf{Q}_- \neq 0$

- ① We look for states with $n_L = -\frac{1}{2}\mathbf{Q}_-^2$.
- ② If $\mathbf{Q}^2 < 0$, we consider an elliptically fibered K3 and 6d heterotic string wrapped on S^1 .
- ③ For $n_L = 0$, winding number w on the S^1 , KK-momentum n_{KK} , and charge \mathbf{Q}_{6d} under the 6d gauge group from the M2-brane, we can write

$$\mathbf{Q} = wC_U + n_{\text{KK}}C_T + \mathbf{Q}_{6d} = \frac{(w + n_{\text{KK}})}{2}C_+ + \frac{(w - n_{\text{KK}})}{2}C_- + \mathbf{Q}_{6d}.$$

- ④ By duality with M-theory in 5d, the heterotic string at excitation n_L is associated to the BPS state obtained by M2-brane on

$$\Sigma = \mathbf{Q} + \frac{n_L}{w}C_T = wC_U + (n_{\text{KK}} + \frac{n_L}{w})C_T + \mathbf{Q}_{6d}.$$

- ⑤ Such BPS states **exist** for $\Sigma \cdot_{\text{K3}} \Sigma \geq 0$,²⁵ implying $n_L \geq -\frac{1}{2}\mathbf{Q}^2$.

²⁵M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th).