Black holes as frozen stars: Regular interior geometry

Tom Shindelman (BGU)

DIP Collaboration Workshop March 23

Based on work w/ R. Brustein, A. J. M. Medved and T. Simhon
[arXiv:2301.09712]

Motivation

Motivation

The problem with gravitational collapse

Classical black holes formed by gravitational collapse:

- Suffer from a singularity [Hawking, Penrose (1970)]
- Cannot be avoided beyond a certain radius (Buchdahl bound)

[Buchdahl (1959)]

Motivation

Why the frozen star solution?

- Is characterized by the eq. of state:

$$p_r = -\rho, \qquad p_\perp = 0$$

- The classical (geometric) proxy of the collapsed polymer

[Brustein, Medved (2017)]

- The frozen star satisfies $r = 2m(r)$ throughout the interior, such that

$$m(r) = 4\pi G \int_0^r dx\, x^2 \rho(x)$$

[Brustein, Medved (2019)]

making every spherical layer a **horizon**

Motivation

Why the frozen star solution?

Assuming a static, spherically sym. spacetime:

$$ds^2 = -f(r)dt^2 + \frac{1}{\widetilde{f}(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

and solving the Einstein eqs. for the condition $\rho = -p_r$:

$$\left(r\widetilde{f}\right)' = 1 - r^2\rho$$

$$\frac{\widetilde{f}}{f}(rf)' = 1 + r^2 p_r$$

leads to

$$f(r) = \widetilde{f}(r) = 1 - \frac{2m(r)}{r}$$

Motivation

Why the frozen star solution?

- The frozen star corresponds to the choice $m(r) = r/2$

$$\boxed{f(r) = \widetilde{f}(r) = 1 - \frac{2m(r)}{r} \quad \mathbf{= 0}}$$

- The corresponding density and pressures:

$$\begin{aligned} \rho &= \frac{1-(rf)'}{r^2} = \frac{1}{r^2} \\ p_r &= -\frac{1-(rf)'}{r^2} = -\frac{1}{r^2} \\ p_\perp &= \frac{(rf)''}{2r} = 0 \end{aligned}$$

Motivation

Why the frozen star solution?

- The eqn. of state of the frozen star is key to its regularity
- Similar exotic matter has been previously studied:
  - EMS tensor at the horizon has the block diagonal form

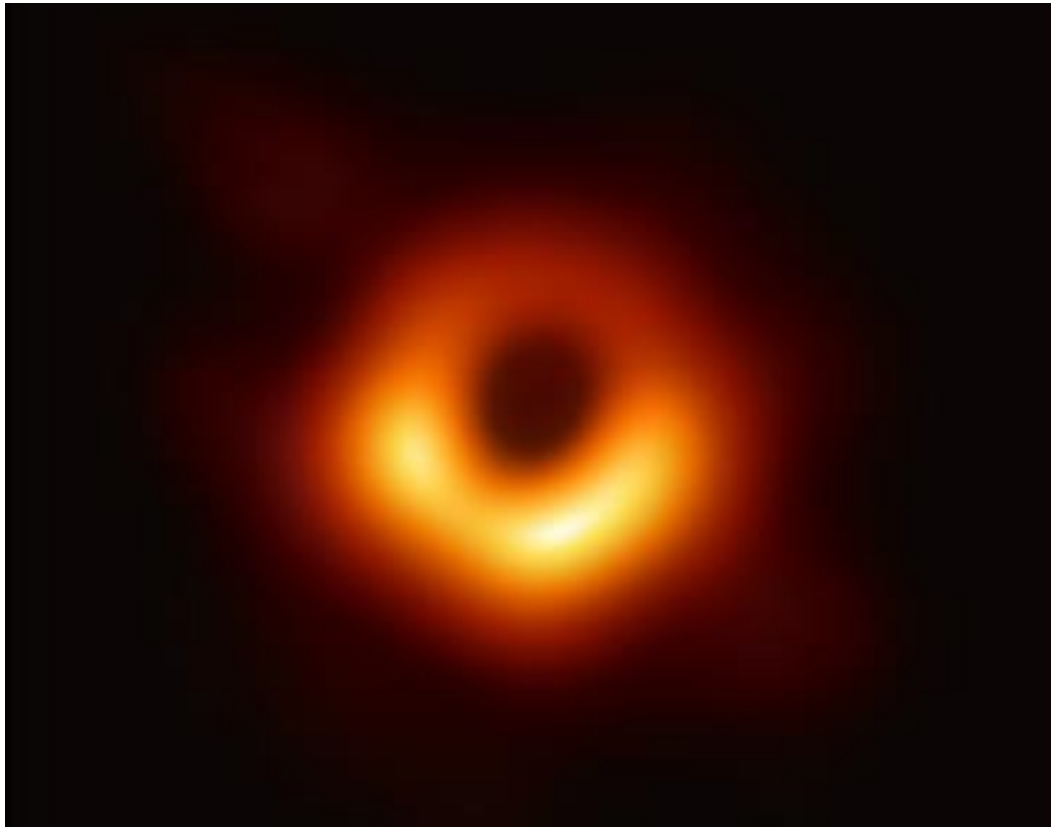
    [Medved, Martin, Visser (2004)]

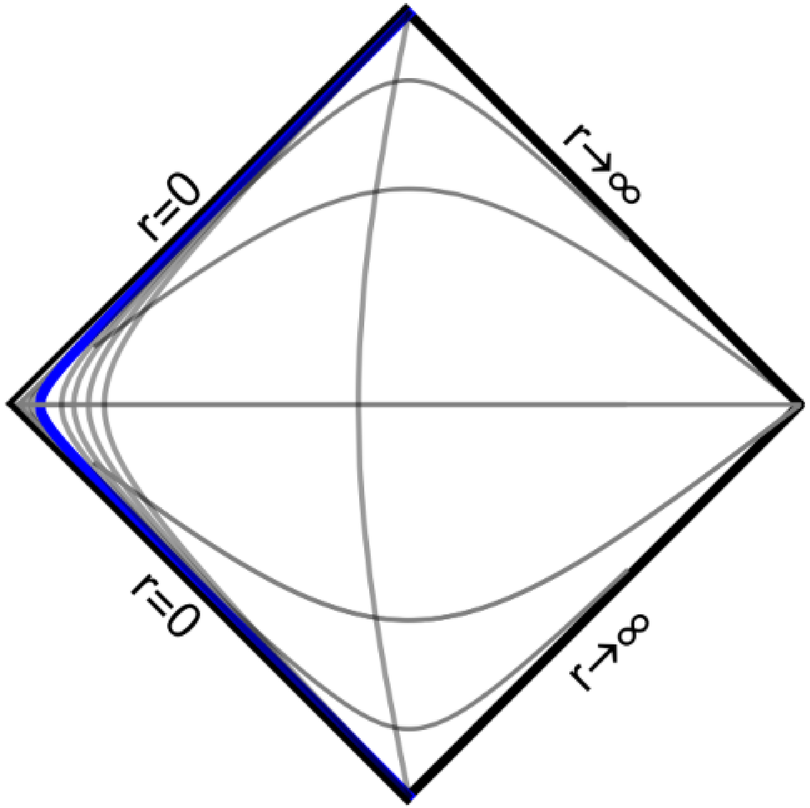
$$p_r = -\rho$$

  - $p = -\rho$ has appeared in other models for BH interiors (Gravastar [Mazur, Mottola (2015)], etc)

    [Chapline, Hohlfeld, Laughlin, Santiago (2001)]

The frozen star model





The frozen star model

Gravitational potential

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2$$

- Replace $f(r) = 0$ with $f(r) = 1 - v^2 \equiv \epsilon \,, \epsilon \ll 1$

[Rabinowitz, Guendelman (1993) (in a Cosmological context!)]

- Coordinates are now manifestly regular

The frozen star model

- The frozen star geometry can be viewed as a spherically sym. collection of rigid 1d strings with tension $1/\alpha'$
- The total mass enclosed in a sphere with radius r is then

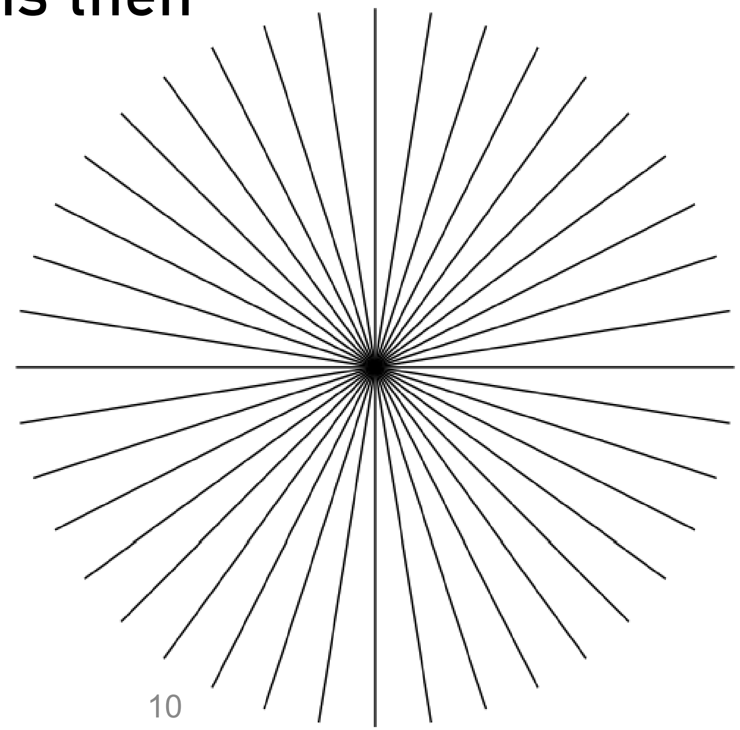
$$m(r) = 4\pi \int_0^r 1/\alpha' = 4\pi/\alpha' \; r$$

- Comparing with

$$m(r) = r/2$$

leads to

$$\alpha' \simeq 8\pi$$



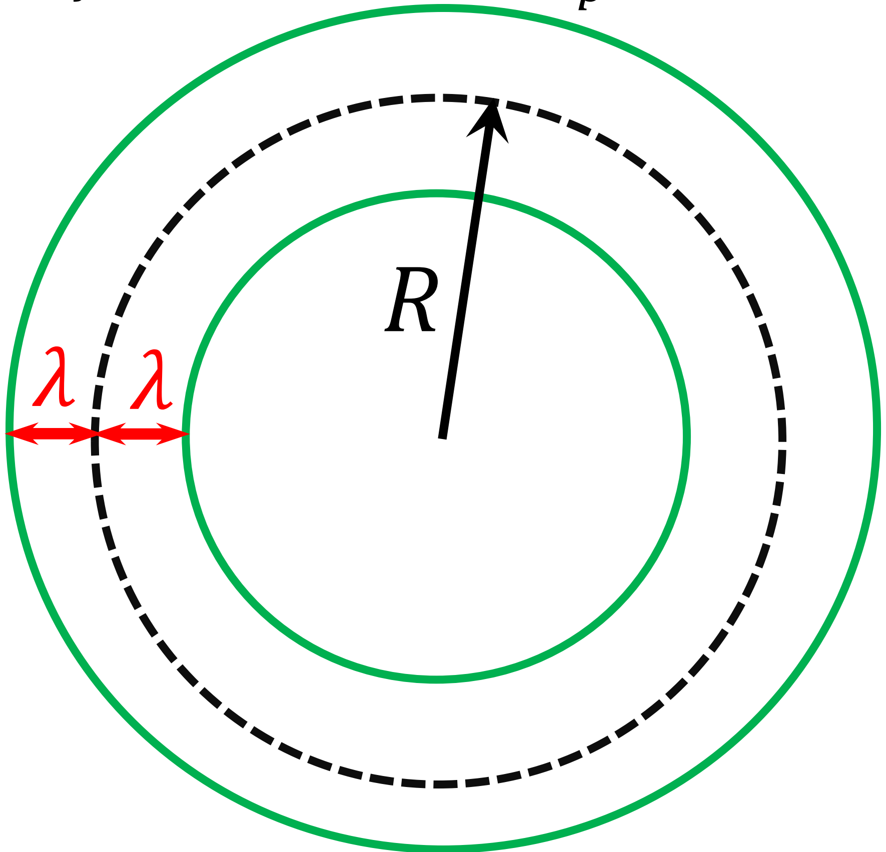
The frozen star model

Regularity of the surface

[Brustein, Medved, Simhon (2022)]

Transitioning from $\rho \neq 0$ *inside* to $\rho = 0$ *outside*

- Define the ***crust***: a layer of thickness $\ell_p \ll 2\lambda \ll R$ on the surface



The frozen star model

Regularity of the surface [Brustein, Medved, Simhon (2022)]

<u>Transitioning from $\rho \neq 0$ *inside* to $\rho = 0$ *outside*</u>

- Define the ***crust***: a layer of thickness $\ell_p \ll 2\lambda \ll R$ on the surface

- Assume key ***symmetries*** of the interior ***persist*** into the crust:

$$\rho = -p_r, \qquad f = \tilde{f}$$

- ***Ansatz***: $f(r)$ is a polynomial expansion in terms of

$$\frac{r-(R-\lambda)}{R} < \frac{2\lambda}{R}$$

- $f(r), f'(r)$ and $f''(r)$ are ***continuous***

The frozen star model

Regularity of the surface [Brustein, Medved, Simhon (2022)]

<u>Transitioning from $\rho \neq 0$ *inside* to $\rho = 0$ *outside*</u>

Assumptions + Matching conditions → Continuous metric throughout crust:

$$
\begin{aligned}
f(x,\lambda) &= \left(\frac{1+4\lambda+5\lambda^2}{4\lambda^2(1+\lambda)^3}\right)(x-1+\lambda)^3 - \left(\frac{(1+3\lambda)(1+5\lambda)}{16\lambda^3(1+\lambda)^3}\right)(x-1+\lambda)^4 \\
&+ \left(\frac{1+3\lambda}{16\lambda^3(1+\lambda)^3}\right)(x-1+\lambda)^5
\end{aligned}
$$

where

$$x = \frac{r}{R}$$

The frozen star model

Regularization of the center [Brustein, Medved, TS, Simhon (2023)]

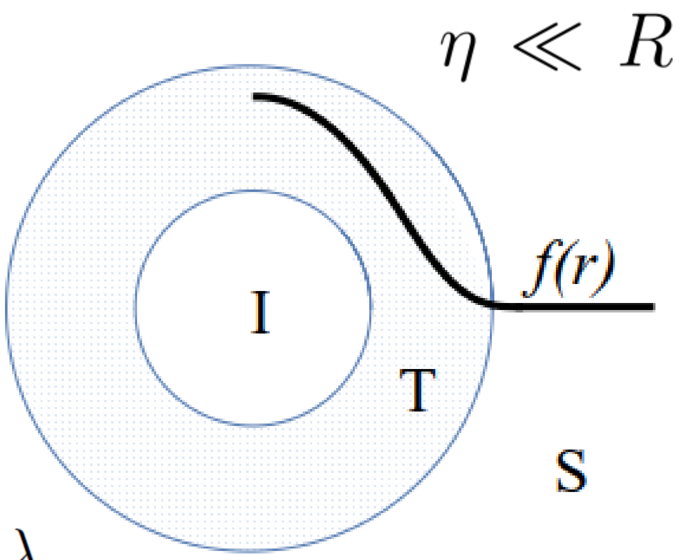
- Similarly, regularization of the divergence $\rho \sim \frac{1}{r^2}$

  will require a transition layer to a ***regular core***

- Region I: The core $r < \eta$

- Region T: Transition layer $\eta < r < 2\eta$

- Region S: Interior of the frozen star $2\eta < r < R - \lambda$

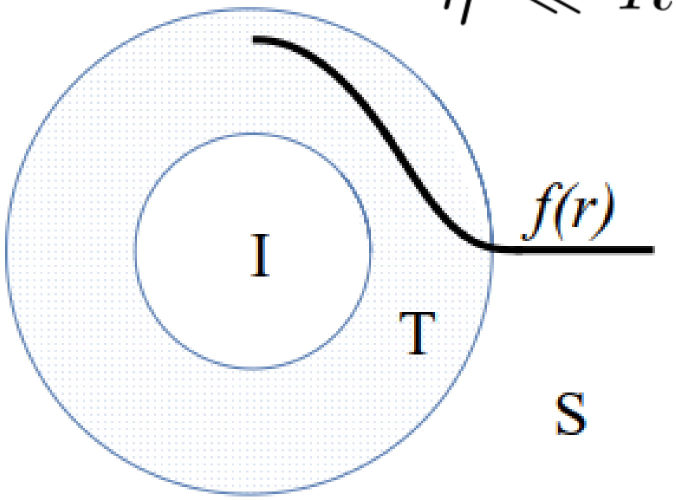


The frozen star model

Regularization of the center [Brustein, Medved, TS, Simhon (2023)]

- Assume:
  - f, f' and f'' are continuous
  - $\rho\left(r=0\right)=\dfrac{B}{\eta^2}$
  - $B>0, \qquad [B]=[length]^{-2}$
  - $\rho_I(B,\eta,r)=-r+\frac{B}{\eta^2}$

$\eta \ll R$



The frozen star model

Regularization of the center [Brustein, Medved, TS, Simhon (2023)]

- Assumptions+Matching conditions lead to

$$f(B,\eta,r) = \begin{cases} 1 - \frac{r^2}{3}\frac{B}{\eta^2} + \frac{r^3}{4}\,, & r < \eta\,, \\ f_T(B,\eta,r)\,, & \eta < r < 2\eta\,, \\ \varepsilon\,, & r > 2\eta\,, \end{cases}$$

$$\frac{6}{5} < B\eta^2 \leq \frac{30}{19}$$

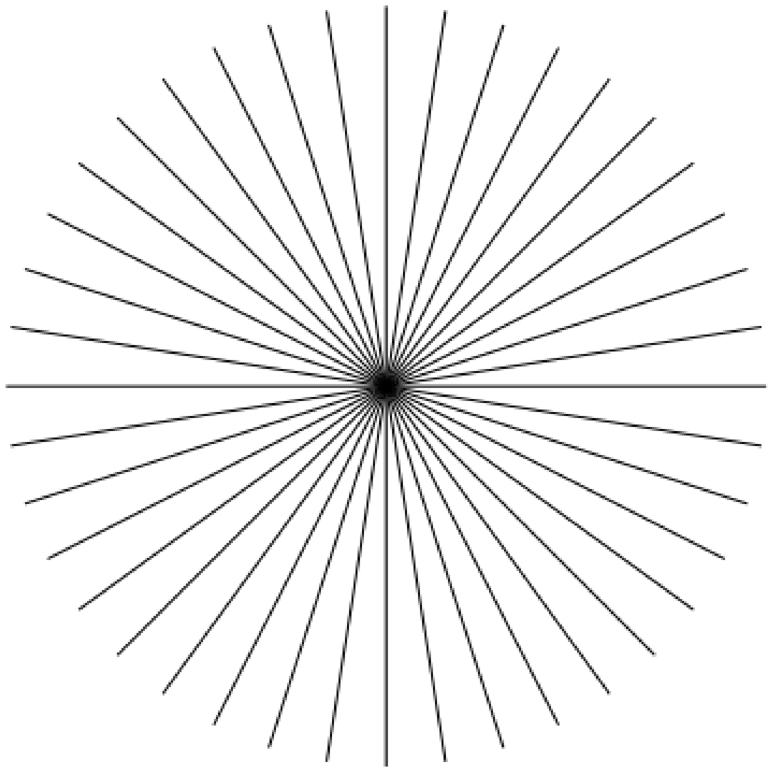
where

$$f_T(B,\eta,r) = \frac{1}{12}(-4B+12) + \frac{(26B-36)(r-\eta)^5}{6\eta^5} + \frac{(-34B+45)(r-\eta)^4}{3\eta^4} + \frac{(100B-120)(r-\eta)^3}{12\eta^3} + \frac{1}{12}\left(-\frac{4B}{\eta^2}\right)(r-\eta)^2 + \frac{1}{12}\left(-\frac{8B}{\eta}\right)(r-\eta)$$

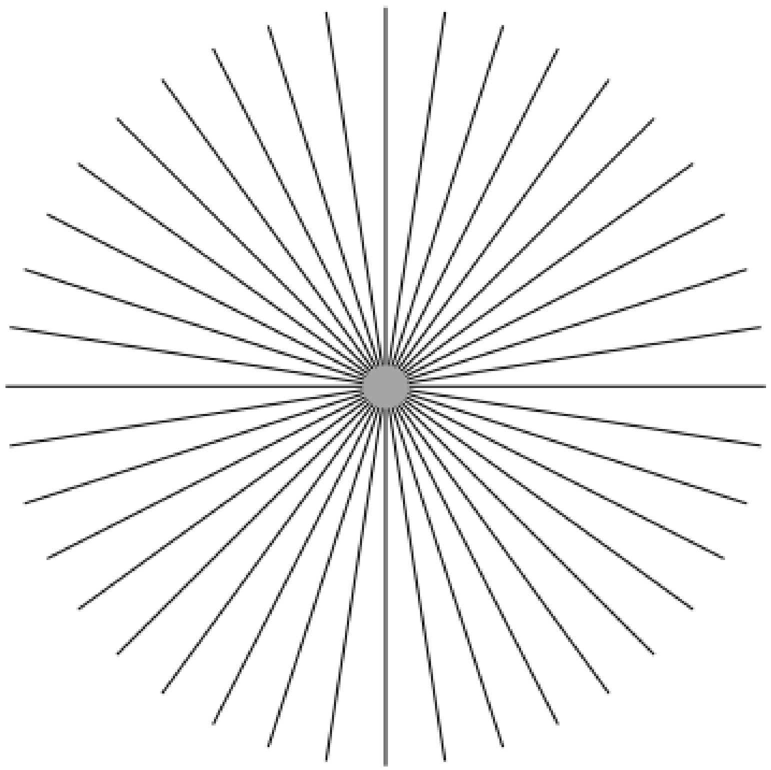
The frozen star model

Regularization of the center [Brustein, Medved, TS, Simhon (2023)]

Unregularized

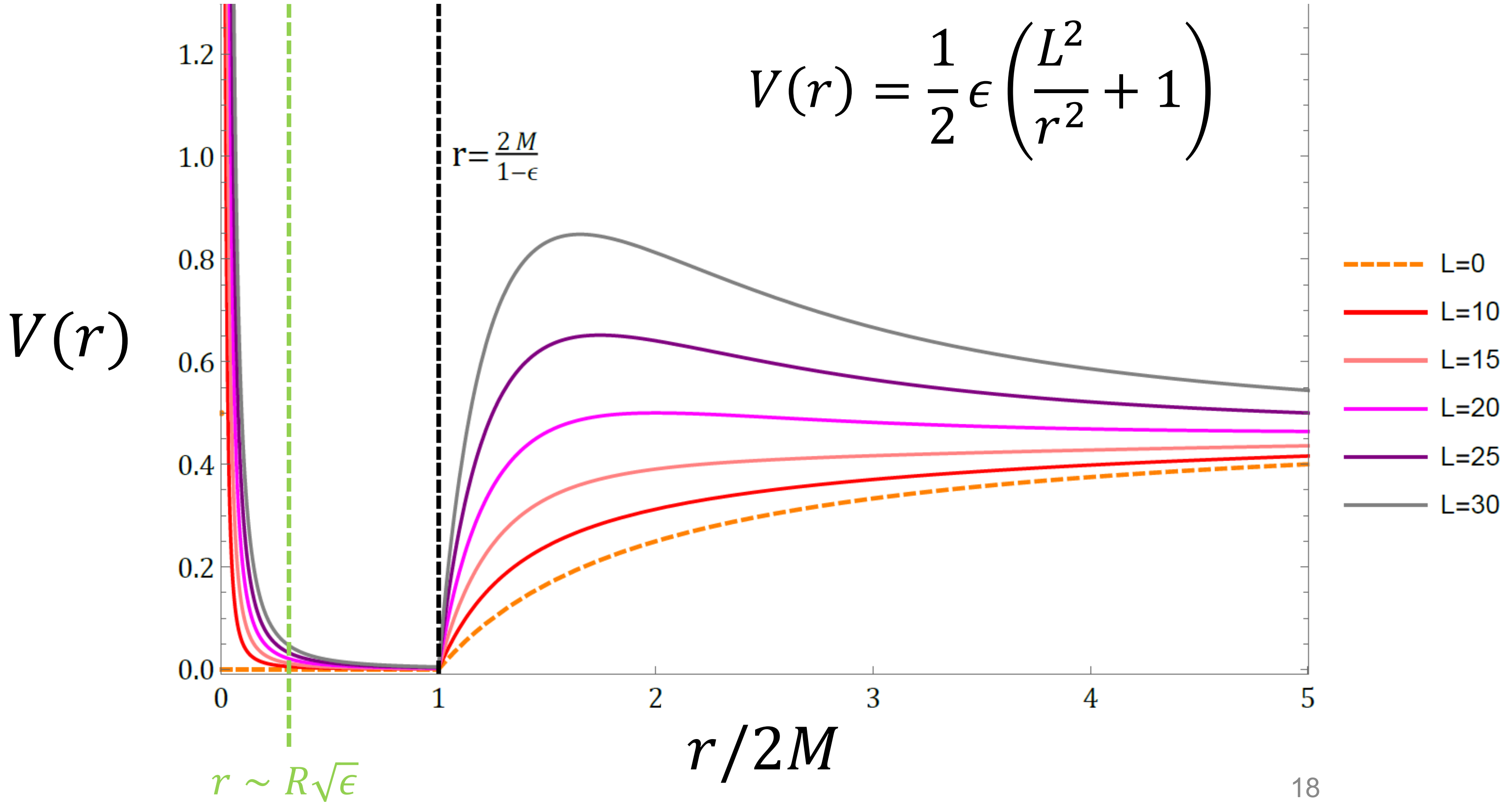


Regularized core



The frozen star model

The gravitational potential experienced by a massive particle



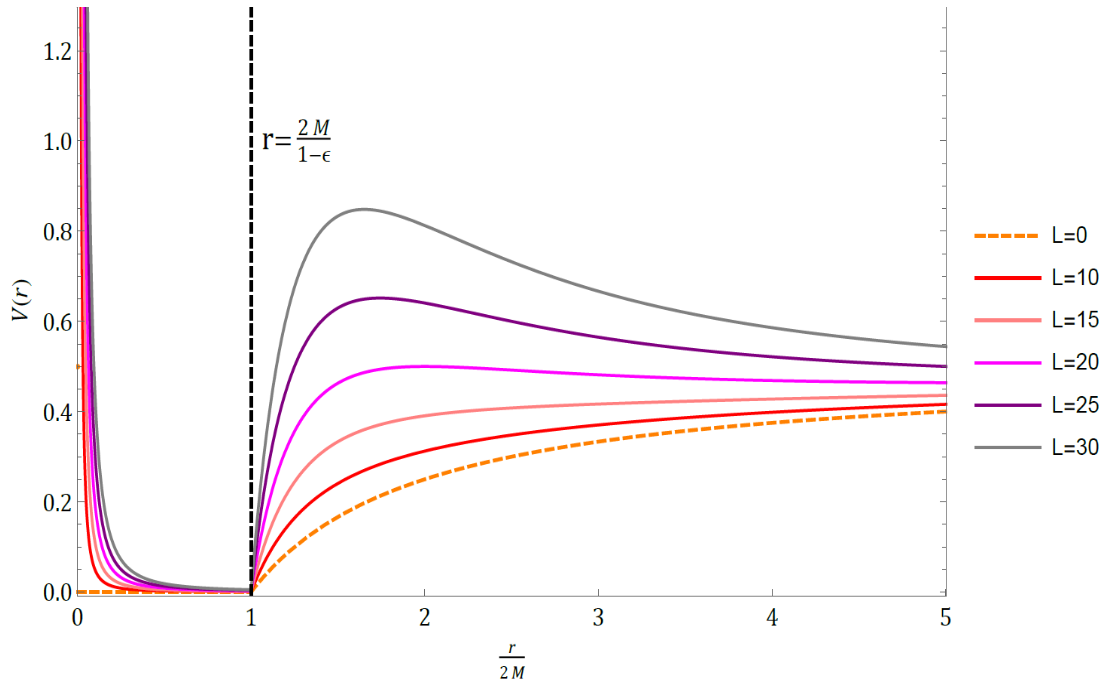
The frozen star model

- $f(r) = \epsilon$ shifts the coordinate radius of the star outwards to

$$R^* = \frac{2M}{1-\epsilon} \approx 2M(1+\epsilon)$$

- Potential barrier for $L > 0$ trajectories deflects matter and light close to $r = 0$:

$$V(r \to 0) \sim \frac{\epsilon L^2}{r^2}$$



The frozen star model

- Purely radial ($L = 0$) trajectories pass thorough the center and reemerge in proper time

$$\Delta\tau = \frac{4M}{(1-\epsilon)\sqrt{E^2 - \epsilon k}} \sim \frac{diameter}{velocity}$$

Where E is the conserved momentum per unit mass,

$$f(r)\frac{dt}{d\tau} \equiv E = const.$$

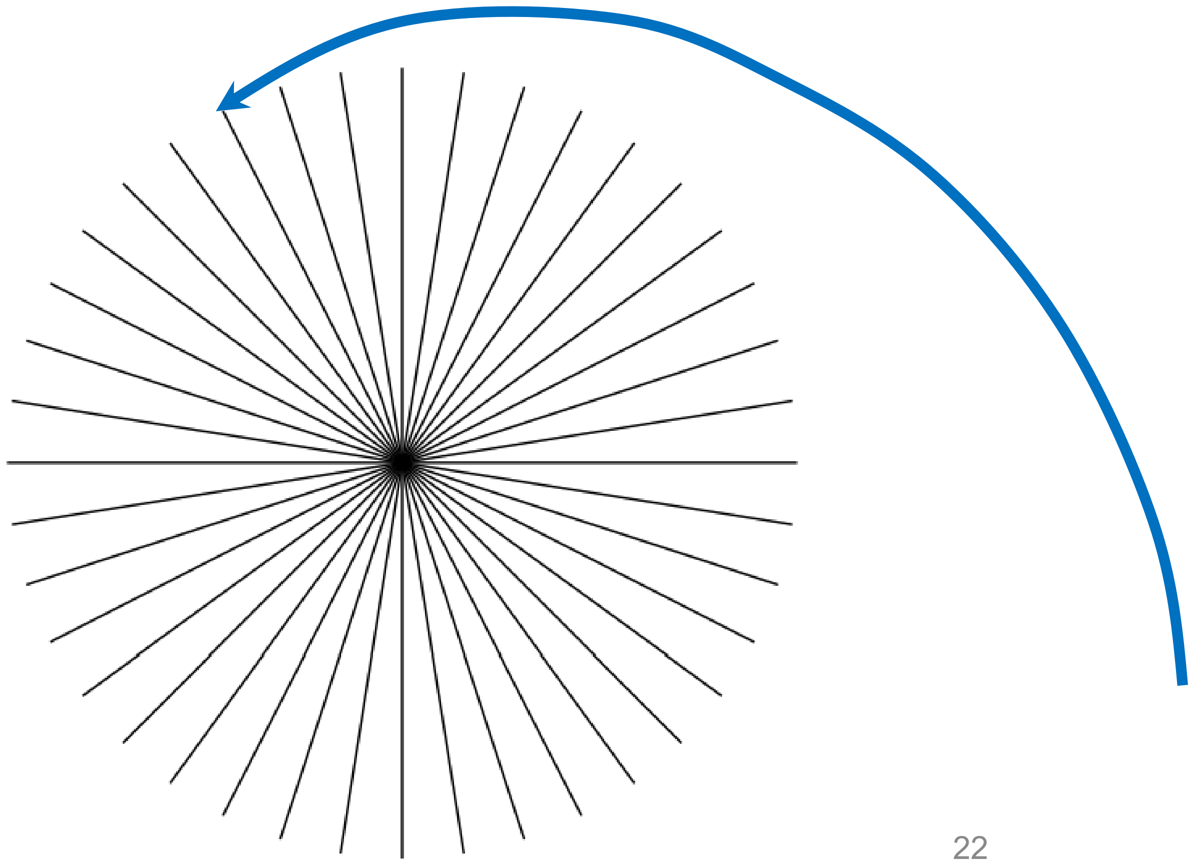
- Very large redshift ($\epsilon \ll 1$) → Frozen star is *essentially black*

Asymptotic coordinate time

$$\Delta t = \frac{4ME}{(1-\epsilon)\epsilon\sqrt{E^2 - \epsilon k}} = \Delta\tau\frac{\mathrm{E}}{\epsilon} \gg 1$$

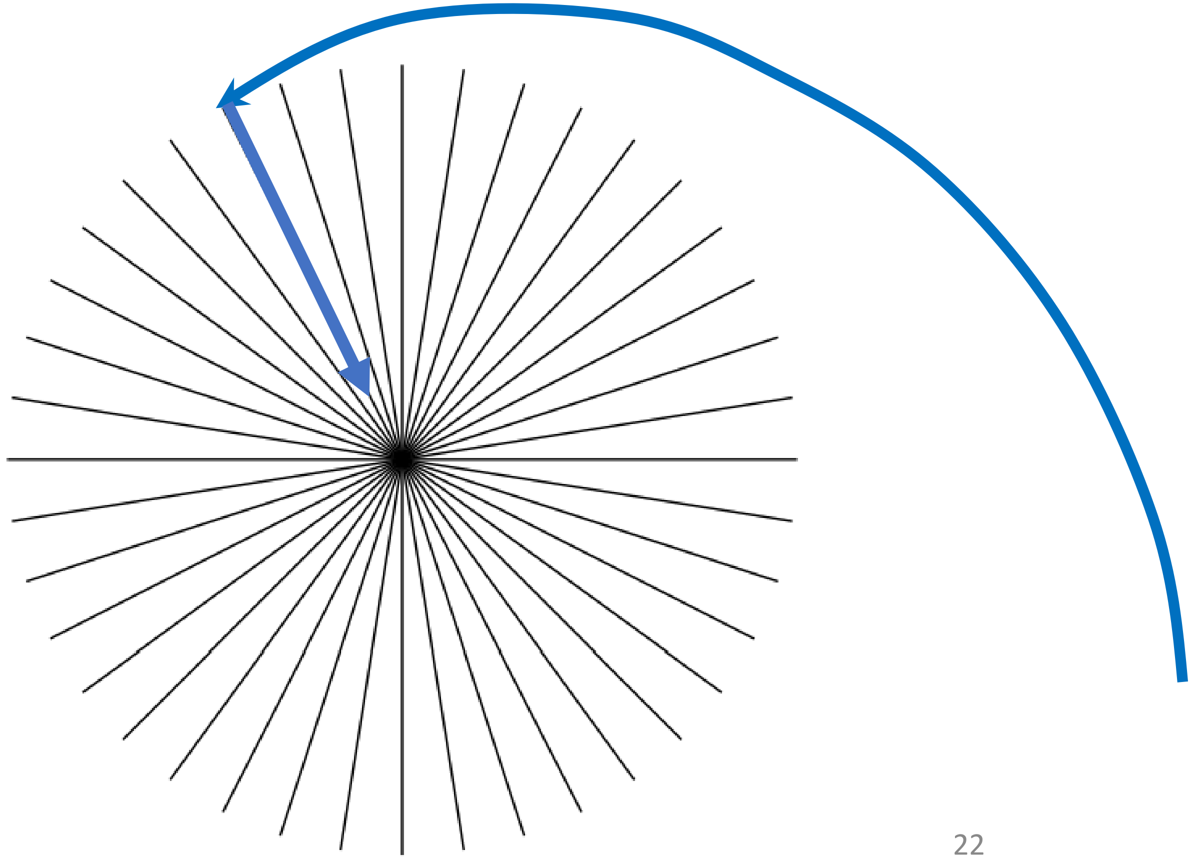
The frozen star model

The trajectory of incoming particles:



The frozen star model

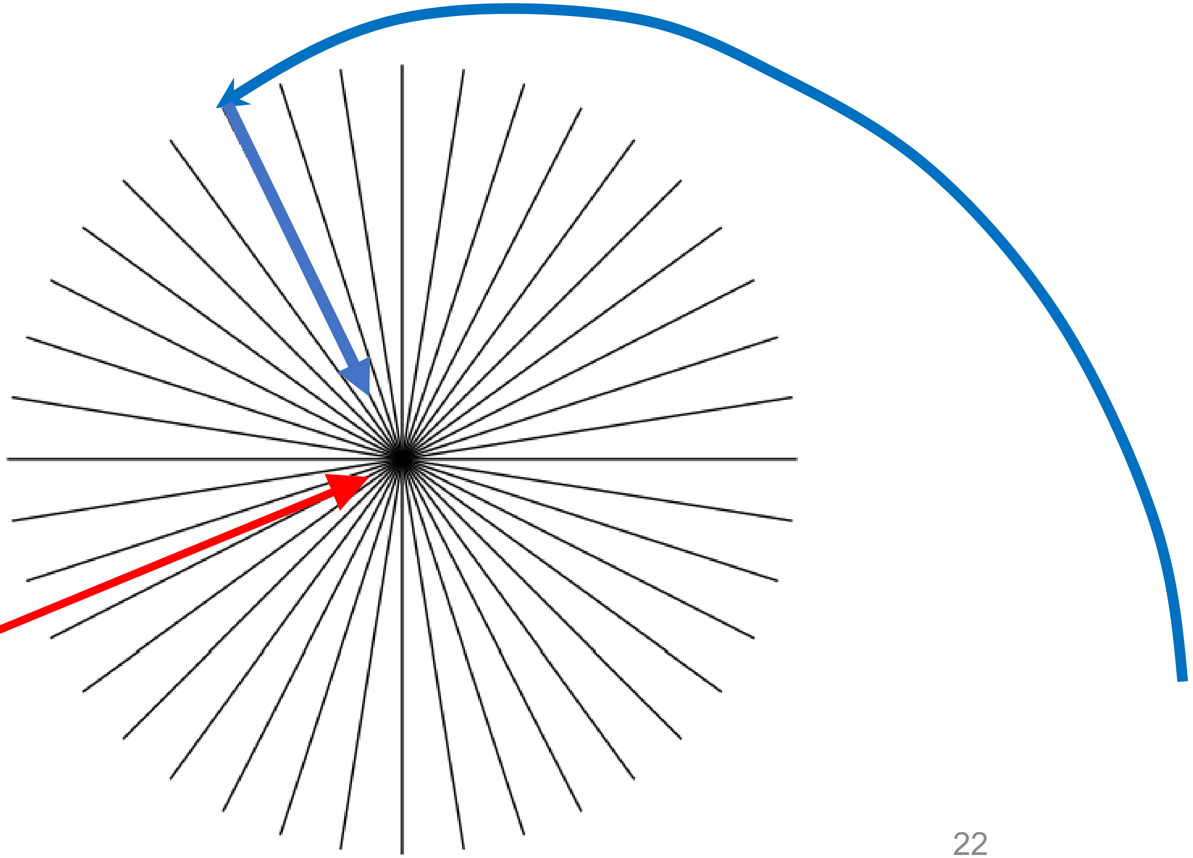
The trajectory of incoming particles:



The frozen star model

The trajectory of incoming particles:

$$V(r \to 0) \sim \frac{\epsilon L^2}{r^2}$$

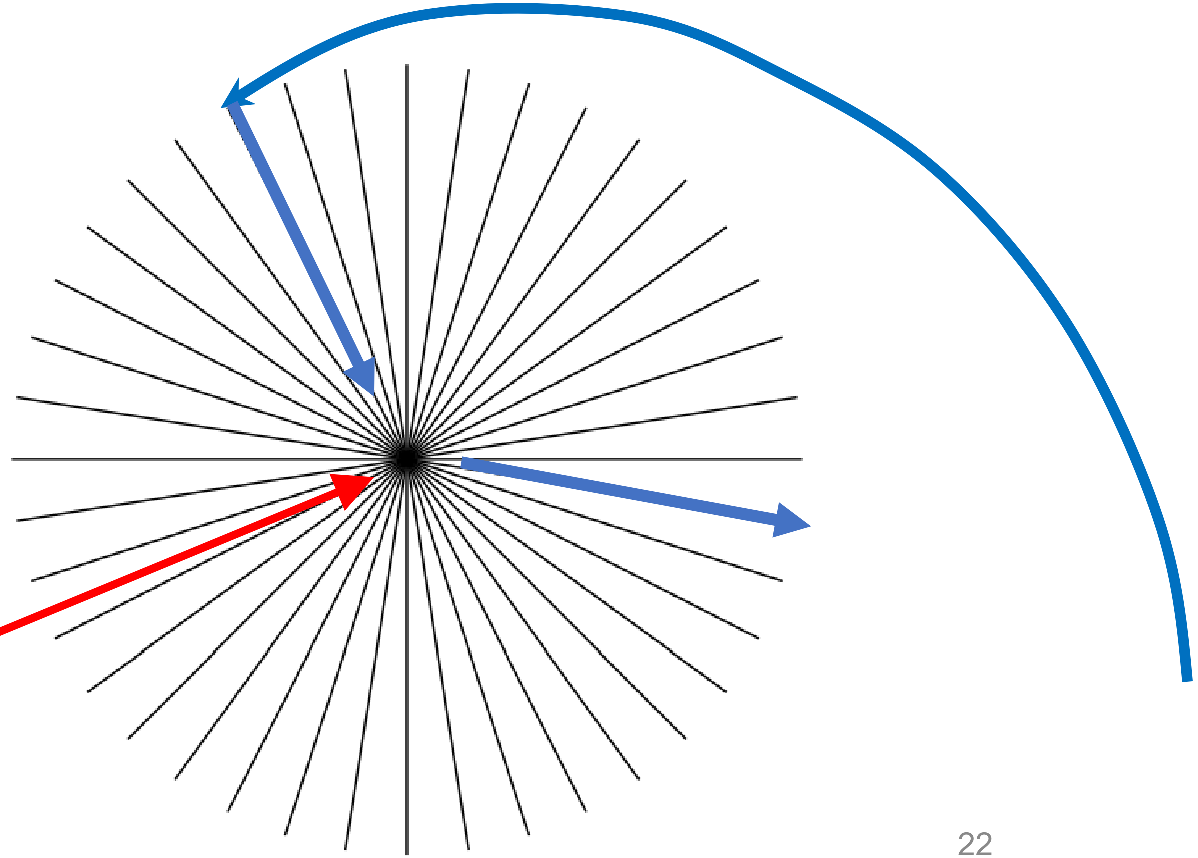


Deflection due to potential barrier close to center

The frozen star model

The trajectory of incoming particles:

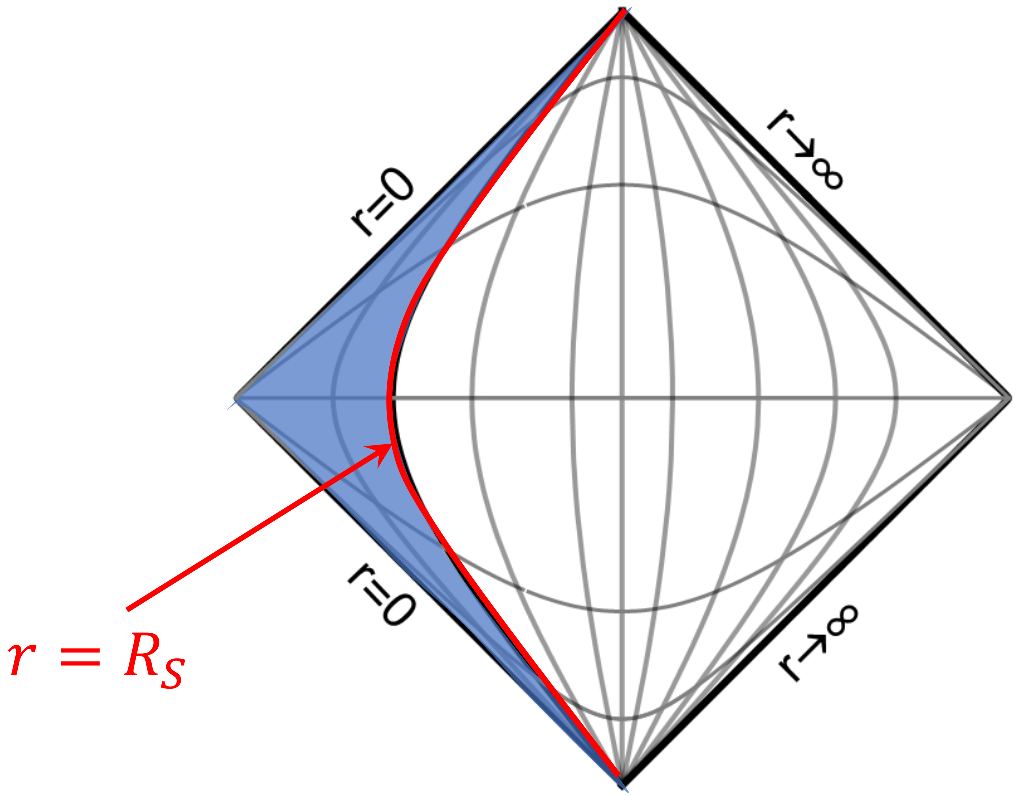
$$V(r \to 0) \sim \frac{\epsilon L^2}{r^2}$$



Deflection due to potential barrier close to center

The frozen star model

Spacetime (Penrose) diagram



The frozen star model

"Kruskal" coordinates - $\epsilon \to 0$ *case*

- When $\epsilon \to 0, \ f(r) \to 0$
- $\det(g)$ is finite, but the redshift diverges
- Solution – look for "Kruskal"-like coordinates

The frozen star model

"Kruskal" coordinates - $\epsilon \to 0$ case

- Kruskal-like coordinate transformation:

$$T = \exp\left[-\frac{r}{\left(1-v^2\right)^{1/2}}\right] \sinh\left(\left(1-v^2\right)^{1/2} t\right)$$
$$R = \exp\left[-\frac{r}{\left(1-v^2\right)^{1/2}}\right] \cosh\left(\left(1-v^2\right)^{1/2} t\right)$$

- The transformed metric:

$$ds^2 = -\frac{1}{R^2 - T^2}\left(dT^2 - dR^2\right) + \frac{1}{4}\left(1-v^2\right)\left(\ln\left(R^2 - T^2\right)\right)^2 d\Omega^2$$

The frozen star model

"Kruskal" coordinates - $\epsilon \to 0$ *case*

$$ds^2 = -\frac{1}{R^2 - T^2}\left(dT^2 - dR^2\right) + \frac{1}{4}\left(1 - v^2\right)\left(\ln\left(R^2 - T^2\right)\right)^2 d\Omega^2$$

- The transformed metric has ***finite*** T, R components
- $t = Const.$ surfaces are straight lines through the origin:

$$\frac{T}{R} = \tanh\left((1 - v^2)t\right)$$

- $r = Const.$ surfaces are hyperbolae:

$$R^2 - T^2 = Exp\left[-\frac{2r}{(1 - v^2)^{1/2}}\right]$$

The frozen star model

Dimensionality when $\epsilon \to 0$

- Kruskal coordinates allow to probe geometry even when $v^2 \to 1$

- Taking $v^2 \to 1$, interior collapses to a single null surface:

$$\underbrace{R^2 - T^2 = Exp\left[-\frac{2r}{(1-v^2)^{\frac{1}{2}}}\right]}_{r=\text{Const. surfaces}} \xrightarrow{v^2\to 1} T^2 = R^2$$

The frozen star model

Dimensionality when $\epsilon \to 0$

- Possible trajectories inside (from Killing equations):

$$\frac{\left(1-v^2\right)^{1/2}}{R^2-T^2}\left[T\frac{dR}{d\lambda}-R\frac{dT}{d\lambda}\right]=\text{Const.}\equiv\mathcal{E},$$

$$\frac{1}{4}\left(1-v^2\right)\left(\ln\left(R^2-T^2\right)\right)\frac{d\phi}{d\lambda}=\text{Const.}\equiv L.$$

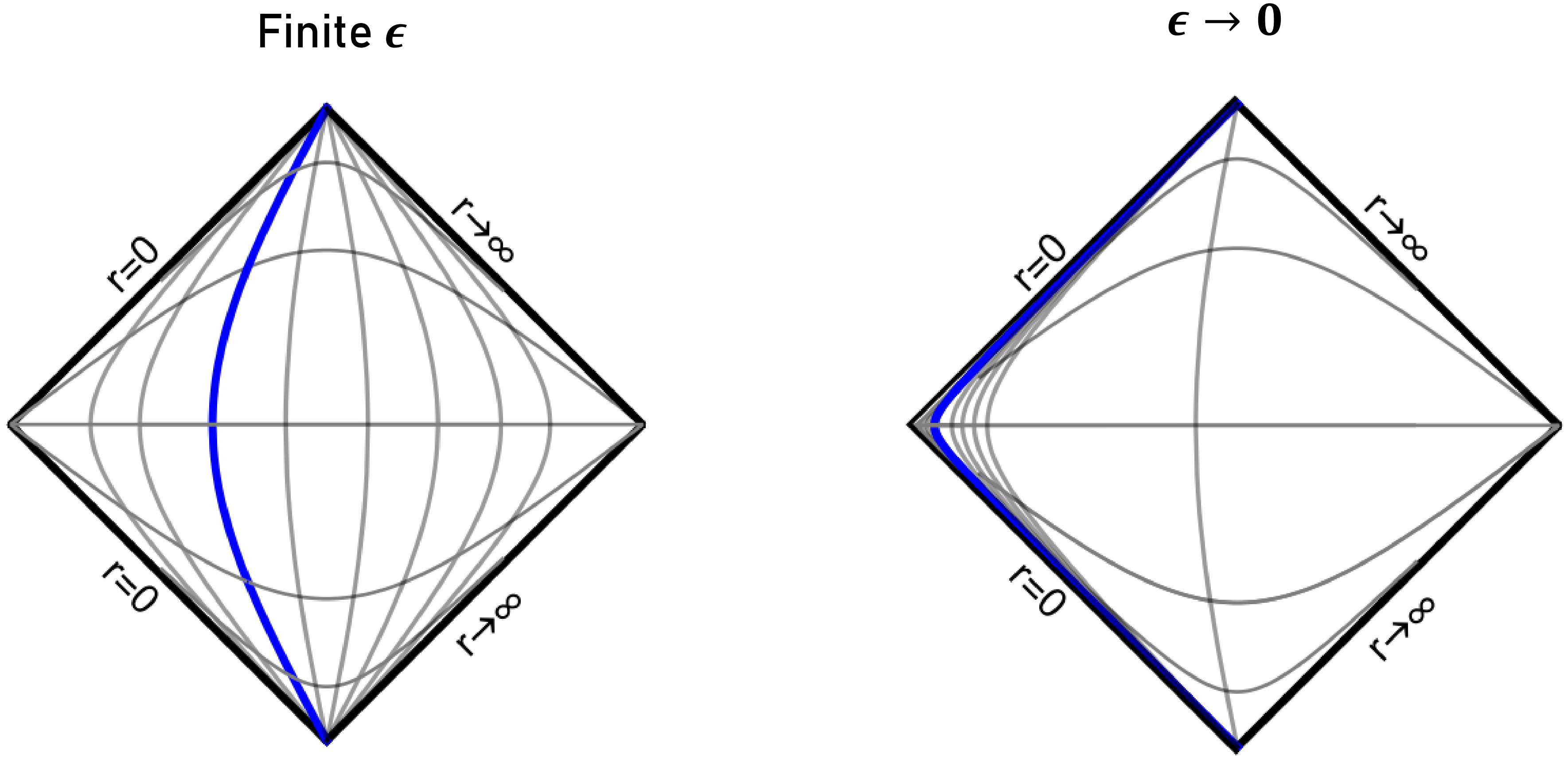
$$\boxed{\textit{Effectively – frozen star is 1+1 dimensional}}$$

- For $v^2 \to 1$ the only remaining trajectories are

$$T = \pm R$$

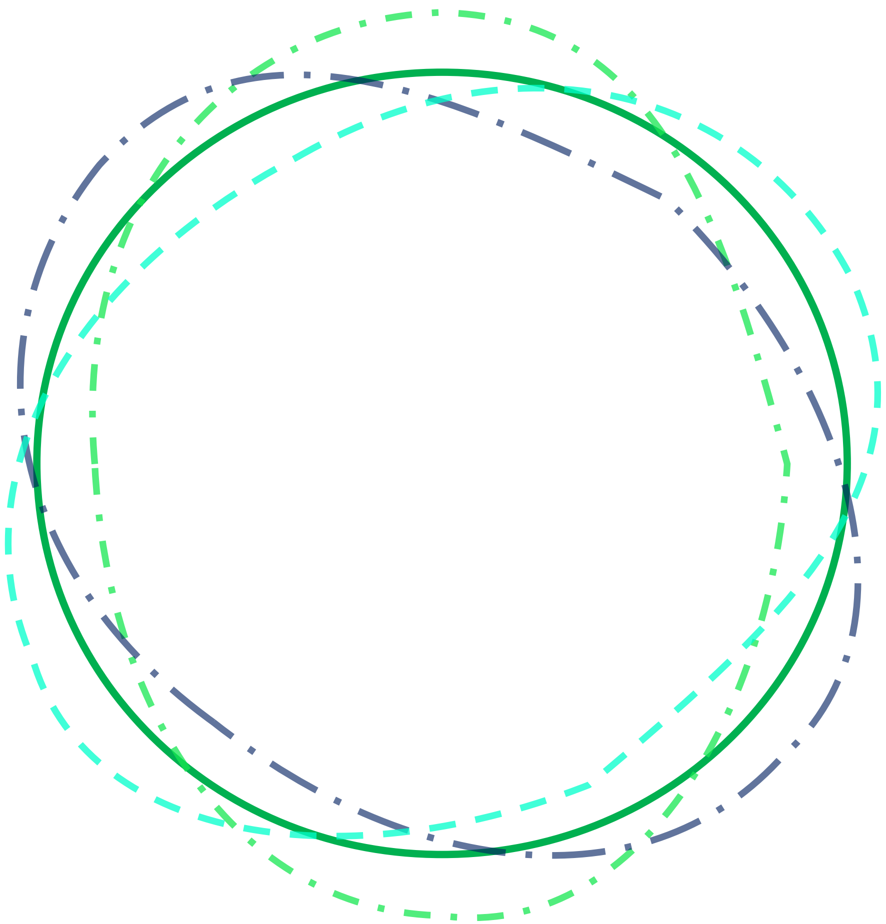
- Motion is permitted <u>only on the horizon</u>– interior is excluded / compactified

The frozen star model



[Brustein, Medved, TS, Simhon (2023), arXiv:2301.09712]

The Oscillation Spectrum



The oscillation spectrum of a frozen star

How to obtain the spectrum of oscillations?

- Frozen star is ultra-stable against perturbations

  [Brustein, Medved, Simhon (2022)]

  $\rightarrow$ trivial spectrum

- Source of ultra-stability is the eq. of state

- Possible solution – slight modification to $\rho = -p_r$

The oscillation spectrum of a frozen star

How to obtain the spectrum of oscillations?

- Frozen star is ultra-stable against perturbations

  [Brustein, Medved, Simhon (2022)]

  $\rightarrow$ trivial spectrum

- Source of ultra-stability is the eq. of state

- Possible solution – slight modification to $\rho = -p_r$

We start here

The oscillation spectrum of a frozen star

How to obtain the spectrum of oscillations?

- Frozen star is ultra-stable against perturbations

[Brustein, Medved, Simhon (2022)]

$\rightarrow$ trivial spectrum

- Source of ultra-stability is the eq. of state

We will show this

- Possible solution – slight modification to $\rho = -p_r$

The oscillation spectrum of a frozen star

- Modify the eq. of state by a small perturbation ($0 < \gamma \ll 1$):

[Brustein, Medved, TS, in preparation]

$$\rho \approx -\left(1 - (a-b)\,\gamma\left(\frac{r}{R}\right)^{b}\right)p$$

- Modification in the eqn. of state is induced by a *modified* metric:

$$g_{tt} = \epsilon + \gamma\left(\frac{r}{R}\right)^{a}$$

$$g^{rr} = \epsilon + \gamma\left(\frac{r}{R}\right)^{b}$$

- $a = 2,\ b = 0$

The oscillation spectrum of a frozen star

- We repurpose an existing framework for oscillations of anisotropic neutron stars [Doneva, Yazadjiev (2012)]
- The frozen star is anisotropic: $p_{\perp} = 0$ but $p_r = -\rho \neq 0$
- Derivations assume Cowling approximation -> No metric perturbations

The oscillation spectrum of a frozen star

- Generic static and spherically symmetric spacetime

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2$$

- Anisotropic fluid

$$T_{\mu\nu} = \rho u_\mu u_\nu + p k_\mu k_\nu + q\left(g_{\mu\nu} + u_\mu u_\nu - k_\mu k_\nu\right)$$

where u^μ is the velocity and k^μ is purely radial such that $u_\mu k^\mu = 0$

- Eqns. of motion: arise from the variation of the energy conservation condition

$$\nabla_\nu \delta T^\nu_\mu = 0$$

- In rest frame: $u^\mu = (u^t, 0{,}0{,}0)$, $k^\mu = (0, k^r, 0{,}0)$

The oscillation spectrum of a frozen star

Equations of motion

- Project in direction *parallel* to u^μ:

$$0 = u^\mu \nabla_\nu \delta T^\nu_\mu = -\nabla_\nu \delta\rho u^\nu - \nabla_\nu \left[\left(\left(\rho+q\right)\delta^\nu_\mu + \sigma k^\nu k_\mu\right)\delta u^\mu\right] \quad \text{(I)}$$
$$-\left(\rho+q\right) a_\mu \delta u^\mu - \nabla_\nu u^\mu \delta\left(\sigma k^\nu k_\mu\right),$$

- Project in direction $P^\mu_\alpha = \delta^\mu_\alpha + u^\mu u_\alpha$ (*perpendicular* to u^μ)

$$0 = \mathcal{P}^\mu_\alpha \nabla_\nu \delta T^\nu_\mu = \delta\left(\rho+q\right) a_\alpha + \left(\rho+q\right) u^\nu \left(\nabla_\nu \delta u_\alpha - \nabla_\alpha \delta u_\nu\right) \quad \text{(II)}$$
$$+ \nabla_\nu \delta q \delta^\nu_\alpha + u^\nu u_\alpha \nabla_\nu \delta q + \mathcal{P}^\mu_\alpha \nabla_\nu \delta\left(\sigma k_\mu k^\nu\right)$$

The oscillation spectrum of a frozen star

Equations of motion

- Define the Lagrangian variation vector

$$\frac{\partial \xi^i}{\partial t} = \frac{\delta u^i}{u^t},\ i = 1, 2, 3 = r, \theta, \phi$$

- Solutions to $\perp$ projection equation have the form

$$\xi_\theta = -\sum_{\ell,m} V_{\ell m}(r,t)\,\partial_\theta Y_{\ell m}(\theta,\phi)$$

$$\xi_\phi = -\sum_{\ell,m} V_{\ell m}(r,t)\,\partial_\phi Y_{\ell m}(\theta,\phi)$$

$$\xi^r = \sum_{\ell,m} e^{-\Lambda}\frac{W_{\ell m}(r,t)}{r^2} Y_{\ell m}$$

The oscillation spectrum of a frozen star

Dynamical equations

Define *oscillatory* modes $W(r,t) = W(r)e^{i\omega t}$, $V = V(r)e^{i\omega t}$:

$$0 = -\omega^2 (\rho + p)\, e^{\Lambda - 2\Phi} \frac{W}{r^2} + \partial_r \hat{\delta p} + \delta\left(\hat{\rho} + \hat{p}\right) a_r + \frac{2}{r}\hat{\delta\sigma}$$

$$0 = (\rho + q)\, e^{-2\Phi} \omega^2 V + \hat{\delta q}$$

The oscillation spectrum of a frozen star

Solutions to leading order in γ:

- $W(r), V(r)$ are power laws:

$$V_{\ell m} \sim V_\ell r^\ell, \quad W_{\ell m} = W_\ell r^{\ell+1}$$

- The oscillation spectrum is:

$$\omega^2 = \frac{\gamma^2}{R^2}\left(2\ell^2 + \ell - 4\right)$$

The oscillation spectrum of a frozen star

<u>Solutions to leading order in γ:</u>

- $W(r), V(r)$ are power laws:

$$V_{\ell m} \sim V_\ell r^\ell, \quad W_{\ell m} = W_\ell r^{\ell+1}$$

- The oscillation spectrum is:

Ultra-stability when $\rho = -p_r$

$$\omega^2 = \frac{\gamma^2}{R^2}\left(2\ell^2 + \ell - 4\right) \Longrightarrow \omega^2\left(\gamma \to 0\right) \to 0$$

Summary and outlook

- Frozen star works well as a BH mimicker

Summary and outlook

- Frozen star works well as a BH mimicker
  - ✓ Regular, well-behaved throughout the interior
  - ✓ Behaves like a Schwarzschild BH from outside

Summary and outlook

- Frozen star works well as a BH mimicker
  - ✓ Regular, well-behaved throughout the interior
  - ✓ Behaves like a Schwarzschild BH from outside
- Is ultra-stable:
  - Must be “defrosted” to obtain an oscillation spectrum
- Next step – rotating frozen stars (more realistic)

Thank you!