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Charge fluctuations and fractional charge of fermions in 1 + 1 dimensions

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Charge fluctuations of solitons with arbitrary fractional fermion number in 1 + 1 dimensions are calculated, generalizing the result of Kivelson and Schrieffer for solitons with fermion number $\frac{1}{2}$. The soliton charge is measured by a sampling function $f(x)$ such that $f(x) \simeq 1$ over a region of width L around the soliton and then falls to zero in a distance l. It is shown that vacuum fluctuations vanish as l^{-1} for large l while the additional fluctuations due to the presence of a soliton vanish as either $\exp(-L/\xi)$ or $\exp(-2\Delta_0 L)$; ξ is the soliton width and Δ_0 is the mass gap of a ground state. This result establishes that the fractional fermion charge is a well-defined observable.

Solitons with fractional charge are of recent interest in both condensed matter physics[1] and quantum field theory.[2,3] The total charge in a given system is integral while the fractional charge is associated with a soliton which is well separated from other solitons in the system. The charge (or fermion number) operator should therefore be defined by

$$\hat{Q}_s = \int dx\, f(x) \psi^\dagger(x) \psi(x) \ , \tag{1}$$

where $\psi(x)$ is the fermion field and $f(x)$ is a sampling function centered around one soliton and excludes all other solitons in the system. We choose therefore $f(x) \simeq 1$ over a region $(-L/2, L/2)$ around the soliton and then $f(x)$ falls to zero in a distance l.[4] The soliton charge is then well defined, namely, the soliton is an eigenstate of the charge operator, if the fluctuations of $\hat{Q}_s$ around its expectation value vanish in the limit of large L and l. It is also important that the fluctuations in the soliton charge, which are in excess of the vacuum fluctuations, decay exponentially with L. All charge moments then decay exponentially and other solitons which are further away then L will not be influenced by these fluctuations.

Kivelson and Schrieffer[5] (KS) have recently calculated the fluctuations in $\hat{Q}_s$ for solitons with charge $\frac{1}{2}$ (Refs. 2 and 6) and have found that the soliton charge is well defined in the above sense. The purpose of this paper is to extend the KS calculations to other models where the soliton charge is arbitrary and establish that it is well defined independently of the soliton detailed shape or the value of its charge. This extension covers solitons of considerable interest such as those with charge $\frac{1}{3}$ (Ref. 1), with charge depending on the couplings[3] or with an irrational spin component.[7]

More specifically, we consider a continuum model of fermions interacting with a classical complex field having an amplitude Δ and phase θ. The system has a discrete set of degenerate ground states with a common amplitude of Δ_0 but distinct values of the phase θ. A soliton configuration corresponds to a space-dependent field with amplitude $\Delta(x)$ and phase $\theta(x)$ which interpolates between two ground states with phase difference $\Delta\theta$. The soliton is assumed to be exponentially localized in a length ξ so that $\Delta(x) \to \Delta_0 + O[\exp(-2|x|/\xi)]$ for $x \to \pm\infty$ while the phase approaches different ground-state values so that $\theta(x) - \theta(-x) \to \Delta\theta + O(e^{-2x/\xi})$ for $x \to +\infty$. The remarkable feature of this configuration is that the fermions acquire a charge $\Delta\theta/2\pi$ (Refs. 1, 3, 8, and 9) which can take any value.

In the calculation below we use the derivative expansion method[9] rather than a coupling-constant expansion.[3] This has the advantage of expanding directly in quantities which are exponentially small far from the soliton center. After presenting the method we first rederive the soliton charge and then evaluate the fluctuations of this charge.

Consider the Hamiltonian

$$H = \int dx\, \psi^\dagger(x) \left[-i\tau_3 \frac{\partial}{\partial x} + \sum_{i=1,2} \Delta_i(x) \tau_i \right] \psi(x) \ , \tag{2}$$

where $\psi(x)$ is a fermion spinor field and τ_i are the Pauli matrices. The fermions couple to classical static fields $\Delta_1(x), \Delta_2(x)$ which relate to the amplitude $\Delta(x)$ and phase $\theta(x)$ through

$$\Delta_1 = \Delta\cos\theta, \quad \Delta_2 = -\Delta\sin\theta \ . \tag{3}$$

The boundary conditions on $\Delta(x), \theta(x)$ represent a soliton as discussed above and complete the definition of the system.

The Hamiltonian (1) can be solved in a power expansion in derivatives of $\Delta_i(x)$. The time-ordered Green's function satisfies

$$\left[i\frac{\partial}{\partial t} + i\tau_3\frac{\partial}{\partial x} - \sum_{i=1,2}\Delta_i(x)\tau_i \right] G(x,t;x',t') = \delta(x-x')\delta(t-t') \ . \tag{4}$$

Define the Fourier transform

$$G(x,t;x',t') = \sum_{\omega,p}\exp[-i\omega(t-t') + ip(x-x')] \times G(x,t;\omega,p) \ , \tag{5}$$

and then the Green's function is solved by the derivative expansion[9]

$$G(x,t;\omega,p) = \sum_{n=0}^{\infty}\left[-G_0(x,t;\omega,p)\left[i\tau_3\frac{\partial}{\partial x} + i\frac{\partial}{\partial t} \right]\right]^n \times G_0(x,t;\omega,p) \ , \tag{6}$$

where

$$G_0(x,t;\omega,p) = \frac{\left[\omega + p\tau_3 + \sum_i \Delta_i(x)\tau_i\right]}{[\omega^2 - E_p^2(x) + i\delta]} \tag{7}$$

and $E_p(x) = +[p^2 + \Delta^2(x)]^{1/2}$. For static $\Delta_i(x)$ the $\partial/\partial t$ in Eq. (6) can be omitted.

The expansion in Eq. (6) is in powers of $(\partial_x \Delta_i)/\Delta^2$ (after ω and p intergrations[9]) and clearly diverges if $\Delta(x) = 0$ at some x. Thus the special case $\Delta_2 \equiv 0$ and $\Delta_1(x) \to \pm\Delta_0$ when $x \to \pm\infty$, which has been studied by KS, is formally excluded. In the following, however, we make explicit use of the derivative expansion only at a distance $x \simeq \pm L/2$ from the soliton center. Since $L/2 \gg \xi$, $\Delta(x)$ is exponentially close to Δ_0, the derivatives are arbitrarily small for L/ξ sufficiently large, and the derivative expansion is well defined. Thus our conclusions are valid even if $\Delta(x)$ is too small (and the derivative expansion diverges) near the soliton center. The derivative expansion is then not valid locally, but still can be used for global properties such as the soliton charge or its charge fluctuations.

The case $\Delta(x) = 0$ for some $x = x_0$ can be considered in a limiting procedure. Since the phase is not defined when $\Delta(x_0) = 0$, various limiting procedures can lead to $\Delta\theta$ values which differ by 2π times integers and the corresponding charges differ by integers. The case considered by KS has ground states with $\theta = 0, \theta = \pi$ and solitons with charges $\pm\frac{1}{2}$ (Ref. 2) which correspond to $\Delta(x)$ approaching zero from two opposite sides.

The charge and current operators are

$$\hat{\rho}(x,t) = \psi^\dagger(x,t)\psi(x,t) \ , \qquad \hat{j}(x,t) = \psi^\dagger(x,t)\tau_3\psi(x,t) \ . \tag{8}$$

Their expectation values, to first order in derivatives, are $\langle j(x,t)\rangle = 0$ and

$$\langle\hat{\rho}(x,t)\rangle = \partial_x\theta(x)/2\pi \ ,$$

where a constant term has been omitted, corresponding to normal ordering of Eq. (8). Using the definition (1) the soliton charge is given by[3,8,9]

$$\langle\hat{Q}_s\rangle = \Delta\theta/2\pi + O(e^{-L/\xi}) \ . \tag{9}$$

To prove Eq. (9) note that an adiabatic switchon of the soliton field from one vacuum together with the conservation law $\partial_t\hat{\rho}(x,t) = -\partial_x\hat{j}(x,t)$ implies that $\langle\hat{\rho}(x,t)\rangle = \partial_x F(x)$, where $F(x)$ is a local function of $\Delta(x), \theta(x)$ and their derivatives. Actually $F(x) = \theta(x)/2\pi +$ derivative terms. Integrating in Eq. (1) by parts gives $\langle\hat{Q}_s\rangle = \int F(x)[\partial_x f(x)]dx$. Now $\partial_x f(x)$ is localized at $x \simeq L/2$, where $\Delta(x), \theta(x)$ are exponentially close to a ground state and the derivative expansion is valid. Thus Eq. (9) is obtained, and the exponentially small corrections come from higher orders in the derivative expansion at $x \simeq \pm L/2$.

We proceed now to our main objective—the evaluation of $\langle\hat{Q}_s^2\rangle$. These fluctuations should vanish in the limit of $L, l \to \infty$ if the limit exists. For $f(x) \equiv 1$ we have $[\hat{Q}_s, H] = 0$, and since there are no degenerate states of different charge the soliton is an eigenstate of $\hat{Q}_s$. Furthermore, for finite L and l, $i[\hat{Q}_s, H] = \int f'(x)\hat{\rho}(x,t)dx$, the integrand being nonzero at $|x| \gtrsim L/2$. It is therefore reasonable to expect that the soliton contribution to the fluctuations is exponentially small. Similar reasoning holds in discrete models.[10]

The required calculation of the fluctuations is considerably simplified if the conservation law is exploited. Consider the charge fluctuations

$$[\delta Q]^2 = \langle\hat{Q}_s^2\rangle - \langle\hat{Q}_s\rangle^2 = \int\int dx\,dx' f(x)f(x')\langle\hat{\rho}(x,t)\hat{\rho}(x',t')\rangle_{t-t'=\eta\to 0+} \tag{10}$$

and the current-current correlation

$$\langle\hat{j}(x,t)\hat{j}(x',t')\rangle = \sum_{\omega,p,\omega',p'}\exp[-i(\omega-\omega')(t-t') + i(p-p')(x-x')]\,\mathrm{Tr}[\tau_3 G(x;\omega,p)\tau_3 G(x';\omega'p')] \ . \tag{11}$$

For $\tau = t' - t < 0$ the poles in ω involve only positive energies at $E_p(x) - i\delta$ while those of ω' involve $-E_{p'}(x') + i\delta$. Therefore replace $\omega - \omega'$ in Eq. (11) by $(\omega - \omega' - i\delta)$ so that the correlation (11) vanishes in the limit $\tau \to -\infty$. For example, for a uniform vacuum one can check that the density-density correlation vanishes as $|\tau|^{-2}$. The conservation law

$$-\partial_\tau^2 \langle \hat{\rho}(x,t)\hat{\rho}(x',t')\rangle = \partial_x \partial_{x'} \langle \hat{j}(x,t)\hat{j}(x',t')\rangle \tag{12}$$

can now be integrated to yield

$$[\delta Q]^2 = \int\int dx\, dx' f'(x) f'(x') \sum_{\omega,p,\omega',p'} \frac{\exp[-i(\omega-\omega')\eta + i(p-p')(x-x')]}{(\omega-\omega'-i\delta)^2} \mathrm{Tr}[\tau_3 G(x;\omega,p)\tau_3 G(x';\omega',p')] \ . \tag{13}$$

Equation (13) is independent of the derivative expansion. Its advantage is that it contains derivatives of $f(x)$ rather than $f(x)$ itself as in Eq. (10). Since $f'(x)$ is exponentially small (or even strictly zero[4]) except for $L/2 < |x| < l + L/2$, the Green's functions in Eq. (13) are needed only where $\Delta(x), \theta(x)$ are exponentially close to a ground state. The derivative expansion is then valid and Eq. (13) can be evaluated by using the zeroth-order Green's functions [Eq. (7)] with corrections of order $\exp(-L/\xi)$. This is a significant reduction in the amount of required calculations.

We stress that the convergence of the derivative expansion is needed only at $|x| \gtrsim L/2$, where $(\partial_x \Delta_i)/\Delta^2 = O(\exp(-L/\xi))$ is small and the expansion indeed converges. As claimed above, our derivation is valid even if the derivative expansion diverges in a region near the soliton center.

Consider first the fluctuations $[\delta Q_0]^2$ in the ground state:

$$[\delta Q_0]^2 = \int dx\, dx' f'(x) f'(x') \sum_{p,p'} \exp[i(p-p')(x-x')] \frac{E_p E_{p'} - pp' + \Delta_0^2}{2E_p E_{p'}(E_p + E_{p'})^2} \ , \tag{14}$$

where $E_p = (p^2 + \Delta_0^2)^{1/2}$. Equation (14) was also derived by KS. For $f(x)$ with a single length scale, e.g., $f(x) = \exp(-x^2/L^2)$, KS show that $[\delta Q_0]^2$ is bounded by a term of order L^{-1}. This can be generalized to an arbitrary $f(x)$ by using Eq. (14) and $|\Delta_0^2 - pp'| \le E_p E_{p'}$,

$$[\delta Q_0]^2 < \tfrac{1}{8}\Delta_0 \int [f'(x)]^2 dx \ . \tag{15}$$

For our $f(x)$ with two length scales this shows that the vacuum fluctuations depend on the length l rather than L, and vanish as l^{-1}. The length l is the range over which $f(x)$ falls from 1 to 0 or the boundary sharpness of $f(x)$. Sharp boundaries increase the vacuum fluctuations, as also known in discrete models.[10]

The fluctuations in the presence of the soliton can now be written in the form

$$[\delta Q]^2 = [\delta Q_0]^2 + \int\int dx\, dx' f'(x) f'(x') \sum_{p,p'} \exp[i(p-p')(x-x')] \frac{\sum_{i=1,2}\Delta_i(x)\Delta_i(x') - \Delta_0^2}{2E_p E_{p'}(E_p + E_{p'})^2} + O(e^{-L/\xi}) \ , \tag{16}$$

where $E_p(x)$ was replaced by E_p with exponentially small corrections. When $x \simeq x' \simeq \pm L/2$ the sum in (16) is also exponentially small so that only terms with $x \simeq -x' \simeq \pm L/2$ remain to be considered. These terms involve the function

$$R(L) = \sum_{p',p} \frac{\exp[i(p-p')L]}{2E_p E_{p'}(E_p + E_{p'})^2} \ . \tag{17}$$

Using the form

$$R(L) = -2 \sum_{\omega,p,\omega',p'} \exp[-i(\omega-\omega')\eta + i(p-p')L][(\omega-\omega'-i\delta)^2(\omega^2 - E_p^2 + i\delta)(\omega'^2 - E_{p'}^2 + i\delta)]^{-1} \tag{18}$$

and the Feynman representation

$$(\omega^2 - E_p^2 + i\delta)^{-1}(\omega'^2 - E_{p'}^2 + i\delta)^{-1} = \int_0^1 d\alpha [\alpha(\omega^2 - E_p^2) + (1-\alpha)(\omega'^2 - E_{p'}^2) + i\delta]^{-1} \tag{19}$$

we obtain

$$R(L) = \frac{L}{4\pi^2\Delta_0} \int_0^1 \frac{d\alpha}{[\alpha(1-\alpha)]^{1/2}} K_1\left[\frac{\Delta_0 L}{[\alpha(1-\alpha)]^{1/2}}\right] \ ,$$

where K_1 is a Bessel function of imaginary argument. Its asymptotic expansion for large L yields the leading term in $R(L)$ as $(4\pi\Delta_0^2)^{-1}\exp(-2\Delta_0 L)$. Thus we find our final result

$$[\delta Q]^2 = [\delta Q_0]^2 + O(e^{-L/\xi}, e^{-2\Delta_0 L}) \ .$$

The excess fluctuations due to the presence of a soliton decay exponentially, as required for the soliton charge to be a well-defined observable. The exponential decay involves both length scales in the problem, the soliton width ξ, and the coherence length Δ_0^{-1}.

Note that with a single length scale, as in $f(x) = \exp(-x^2/L^2)$, $f'(x)$ behaves like x/L^2 in the soliton vicinity ($|x| < \xi << L$) and the excess fluctuations then decay as a power of L. When $f(x)$ has two length scales such that $f'(x)=0$ or is exponentially small for $|x| < L/2$,[4] then the excess fluctuations decay exponentially and are independent of l. KS define $L/2$ to be at the center of the tail, i.e., at the extrema of $f'(x)$. In that case if $l \gtrsim L$, $f'(0)$ is not exponentially small, and they need the condition $l << L$. We defined $L/2$ to be where the tail starts so that for any l, $f'(x)$ is zero (or exponentially small) for $|x| < L/2$.

In conclusion, we have shown that the fluctuations in the vacuum charge vanish as l^{-1} for $l >> \Delta_0^{-1}$ and the excess fluctuations from the presence of a soltion decay exponentially for $L >> \Delta_0^{-1}$, ξ. These conclusions are in agreement with those of KS and generalize them to solitons with arbitrary shape or charge.

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$$f'(x)=0$$

for

$$|x| \leq L/2 \ , \text{or} \ |x| \geq l+L/2 \ ,$$

$$f'(x) = -\operatorname{sgn} x C \exp-[(|x|-\tfrac{1}{2}L)^{-2}+(\tfrac{1}{2}L+l-|x|)^{-2}]$$

for

$$L/2 \leq |x| \leq l+L/2 \ .$$

C is chosen such that $\int_{-\infty}^{0} f'(x)dx = 1$.

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