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Self-induced transparency and the soliton lattice

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The propagation of a step-function light pulse in a nondamping resonant medium is studied. Numerical analysis and conservation laws show that the asymptotic solution cannot be described by a single soliton-lattice-type solution. As the pulses propagate in space they become narrower and with larger amplitude, but the time separation between pulses remains constant. This result is explained by an asymptotic area theorem which restricts the time average of the electric field to be equal to that of the input field.

I. INTRODUCTION

The propagation of coherent optical pulses in a resonant medium has led to the remarkable phenomenon of self-induced transparency.[1–3] An input pulse of finite duration in time decomposes asymptotically into a number of individual steady-state pulses. Each pulse is a soliton with the associated electric field behaving as a hyperbolic secant; the number of these pulses is fixed by the area theorem.[1–3]

Here we consider the situation of a step-function imput pulse, with infinite duration in time. Since pulses are constantly generated, the asymptotic solution cannot decompose into well-separated solitons, and the usual area theorem is not useful.

Instead, it is conceivable that "soliton-lattice" solutions may be an asymptotic description of this situation. A soliton-lattice solution[4–6] corresponds to an electric field proportional to $dn\,[(t-z/v)/\tau;\lambda]$, where dn is a Jacobian elliptic function[7] with parameter λ $(0<\lambda<1)$, v is the velocity of the soliton lattice, and τ measures the time width of each soliton. The periodicity of $dn\,(u;\lambda)$, where $u=(t-z/v)/\tau$, is $2K(\lambda)$, and K is the complete elliptic integral[7]; it diverges when $\lambda\rightarrow 1$ and then each pulse is a well-separated sech(u) soliton.

The motivation for this study is threefold. First there is a renewed interest in soliton lattices in other areas of physics, in particular in condensed matter physics.[8,9] Secondly, recent experiments[10] on a relatively dense medium of ReO_4^- impurities in a KBr or KI matrix with long pulses showed generation of a sequence of many pulses. It is not yet clear however, if self-induced transparency is indeed responsible for this phenomenon.

The final motivation is the article by Crisp,[6] who investigated the soliton-lattice solution in detail. Crisp claimed that a step-function input evolves asymptotically into a soliton-lattice solution with $\lambda=\frac{4}{5}$. The clue for this peculiar value of $\lambda=\frac{4}{5}$ was not sufficiently clear and led us to repeat Crisp's calculation. We find that going to longer times or further in space then Crisp did, a soliton lattice is not the asymptotic solution. As the pulse train propagates in space, each pulse becomes narrower and with larger amplitude; this corresponds to some local "λ" which gets closer to 1 as function of space.

In addition we use analytic conservation laws for energy and momentum, which show that a propagating soliton lattice cannot be the only asymptotic behavior. Instead, however, we show an asymptotic area theorem which fixed the time average of the electric field to be equal to that of the input field. This implies for a $dn\,(u;\lambda)$ solution that the area of each pulse is 2π, in close agreement with the numerical results.

II. NUMERICAL SOLUTION

The self-induced transparency equation of a nondamped medium, without inhomogeneous broadening and at resonance reduced to a single differential equation[1–3] for a phase variable $\theta(z,t)$

$$\ddot{\theta}(z,t)+c\dot{\theta}'(z,t)=-\alpha c\sin\theta(z,t)\ , \tag{1}$$

where the dot is $\partial/\partial t$ and the prime is $\partial/\partial x$; c is the velocity of light and $\alpha=2\pi n\omega\mu^2/(\text{c}\hbar)$, where n is the density of resonating atoms, μ their dipole moment, and ω the resonance frequency. The electric field envelope is given by

$$\mathcal{E}(z,t)=\dot{\theta}(z,t)\mu/\hbar\ . \tag{2}$$

Equation (1) is studied with the following boundary conditions.

(a) The input pulse is a step function

$$\mathscr{E}(0,t)=\begin{cases}\mathscr{E}_0, & t>0\\ 0, & t<0\end{cases}, \tag{3}$$

where $z=0$ is the boundary between the medium ($z>0$) and the vacuum ($z<0$).

(b) The system is initially in its ground state $\theta=0$, i.e., it is an attenuator,

$$\theta(z,t)=0 \quad \text{for } t<z/c\ . \tag{4}$$

In terms of the dimensionless variables $\epsilon=\mathscr{E}/\mathscr{E}_0$

$$x=z\alpha\hbar/(\mu\mathscr{E}_0)\ ,$$
$$y=(t-z/c)\mu\mathscr{E}_0/\hbar\ , \tag{5}$$

Eq. (1) can be written as

$$\frac{\partial\epsilon}{\partial x}=-\sin\theta\ ,$$
$$\frac{\partial\theta}{\partial y}=\epsilon\ , \tag{6}$$

with the boundary conditions in the relevant range $y\geq 0$, $x\geq 0$,

$$\epsilon(0,y)=1\ ,$$
$$\theta(x,0)=0\ . \tag{7}$$

Following Crisp[6] we study this system by successive substitution in the integral equations

$$\epsilon(x,y)=1-\int_0^x \sin(x',y)dx'\ ,$$
$$\theta(x,y)=\int_0^y \epsilon(x,y')dy'\ , \tag{8}$$

starting from some initial functions, e.g., $\epsilon^{(0)}(x,y)=1$, $\theta^{(0)}(x,y)=0$, Eq. (8) generates the first-order functions, which in turn generate the second-order functions, etc. The solution is then the limit of this set of functions, provided the limit exists.

The number of required iterations increases rather fast with the range of (x,y) being solved; for the results below we needed up to 120 iterations to achieve convergence of at least six significant digits. The results are independent of which initial function is used, since the boundary conditions are sufficient to guarantee a unique solution.

The results are shown in Figs. 1–5. In Figs. 1 and 2 the results for small x and $y\leq 200$ are shown. The results of Crisp[6] for $x=1,2,3$ and $y\leq 100$ are similar to ours, and seem to indicate a steady-state solution. This solution has a periodicity of $\Delta y\simeq 6.4$ and from its amplitude Crisp concluded that it is a $dn(u;\lambda)$ with $\lambda=\frac{4}{5}$. However, by considering also $100<y<200$ it is seen that there is an additional longer periodicity and the solution cannot be represented by a $dn(u;\lambda)$ which has a single periodicity.

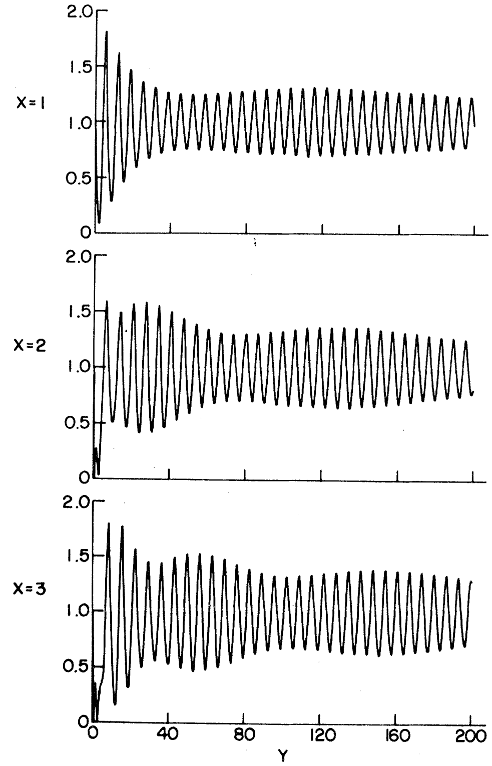

FIG. 1. The electric field $\epsilon(x,y)$ which solves Eq. (6) with the boundary conditions Eq. (7), for $x=1,2,3$ with $y\leq 200$.

The results for larger values of x in Fig. 3 show even more significant changes. As x increases, the pulse amplitude becomes larger, and each pulse becomes narrower. This corresponds to $dn(u;\lambda)$ functions whose parameter λ is $\sim\frac{4}{5}$ at $x=1$ and increases with x to $\lambda\sim 0.95$ at $x=11$. The functions $dn(u;\lambda)$ are exact solutions of Eq. (1) only for a constant λ; the "x-dependent λ" is just a qualitative way of describing the solution.

The narrow peak near $y=0$ with $\epsilon(x,0)=1$ represents a short pulse propagating with velocity c, much faster than the pulse train. It is a result of the sudden turn-on of the input pulse at $t=0$, i.e., the medium cannot respond sufficiently fast to slow down the initial edge of the input pulse. This effect can be eliminated by turning on the input pulse more slowly.

Figures 4 and 5 show the results as function of x

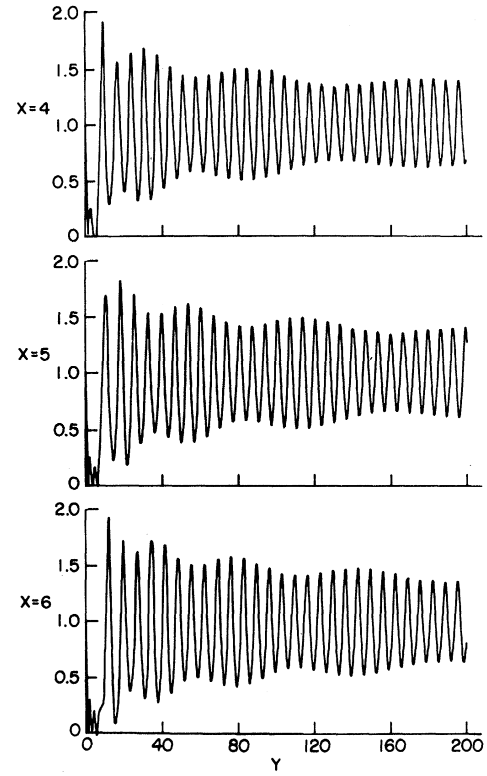

FIG. 2. Same as Fig. 1 with $x=4,5,6$.

for a given y. Since $y=\omega_R t-(\omega_R^2/\alpha c)x$, where $\omega_R=\mu\mathcal{E}_0/h$ is the Rabi frequency, for $\omega_R^2 \ll \alpha c$ a given y is almost a given t, if x is not too large [experimentally[1,10] $\omega_R^2/(\alpha c)=10^{-2}-10^{-4}$].

The results in Figs. 4 and 5 demonstrate again that the pulses tend to separate as x increases, i.e.,

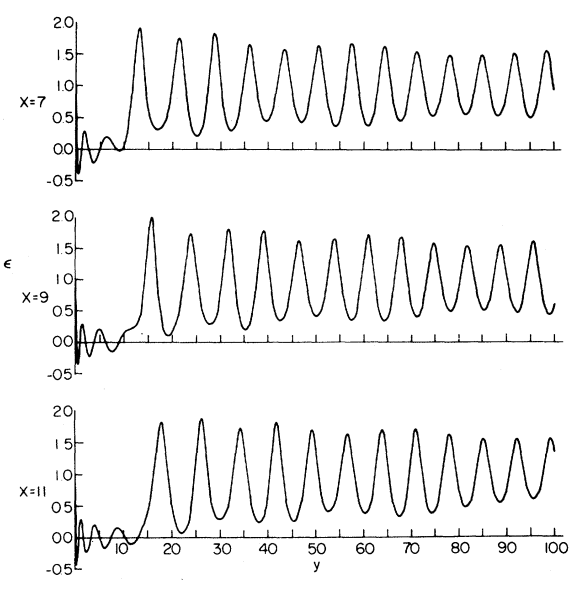

FIG. 3. Same as Fig. 1 with $x=7,9,11$, with $y\leq 100$.

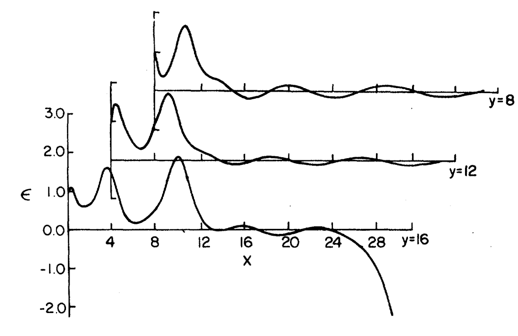

FIG. 4. $\epsilon(x,y)$ as function of x with $y=8,12,16$.

the pulses become narrower with larger amplitude. For $x\sim 20$ there is an additional feature—the pulse train propagates by a $(0-\pi)$-like pulse in its front which generates an additional 2π pulse to the pulse train.

Figure 6 shows the position of the pulse train front x_{th}, defined as the maximal x for which $\epsilon=1$, as function of y. The curve shows some oscillations which mean that the pulse train does not move uniformly with a constant velocity. The oscillations are however rather small, and the mean slope of the curve defines a velocity $U=x_{\text{th}}/y \simeq 0.85$. Using Eq. (5) the corresponding velocity in real space is

$$v=c[1+\alpha c/(\omega_R^2 U)]^{-1}\ . \tag{9}$$

For $\alpha c \gg \omega_R^2$ this yields $v \ll c$. The condition $\alpha c \gg \omega_R^2$ means that, the energy stored in the resonating atoms is much larger than the electromagnetic energy.

Although the solutions cannot be described by a single $dn(u;\lambda)$ solution, they do have a uniform characteristic. This is the periodicity, or the distance between successive peaks, which is in the range 6.3−6.5. This will be explained in Sec. III in terms of an asymptotic area theorem.

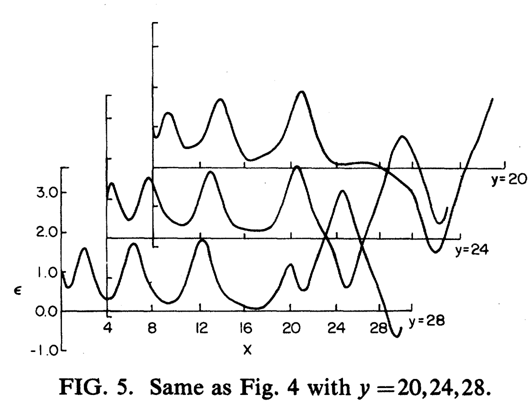

FIG. 5. Same as Fig. 4 with $y=20,24,28$.

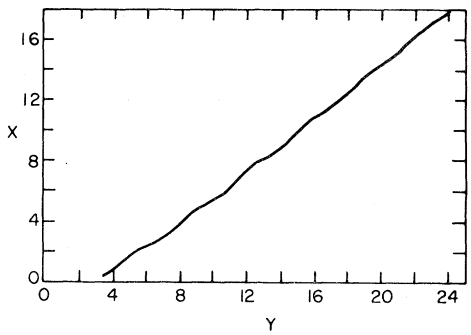

FIG. 6. The threshold value of x, defined as the maximal x for which $\epsilon(x,y)=1$ as function of y. The average slope gives the velocity of the pulse-train front, $U \simeq 0.85$.

III. CONSERVATION LAWS AND THE SOLITON LATTICE

Conservation laws are an analytic property of the solutions of Eq. (1) and are very useful in determining asymptotic solutions.[2,11] Equation (1) has an infinite number of conservation laws[2,12]; for our purpose it is sufficient to use only two of these which correspond to energy and momentum conservation.

Conservation laws can be derived by noting that Eq. (1) is the Euler-Lagrange equation of the following Lagrangian density:

$$L(z,t)=\tfrac{1}{2}\dot{\theta}^2+\tfrac{1}{2}c\dot{\theta}\theta'+\alpha c(\cos\theta-1)\ , \qquad (10)$$

where $\theta=\theta(z,t)$. Invariance under time or space translations leads to conserved energy and momentum.

For our system with $z>0$, the energy is

$$E(t)=\int_0^\infty dz\left[\tfrac{1}{2}\dot{\theta}^2+\alpha c(1-\cos\theta)\right]\ . \qquad (11)$$

This is the total energy in the system, in units of $\hbar^2/(2\pi\mu^2)$. The first term is the electromagnetic energy $\mathcal{E}^2(z,t)/4\pi$, while the second term is the energy stored in the resonating two-level system $n\hbar\omega(1-\cos\theta)$, since $\cos\theta$ is the degree of inversion. Note that the choice of signs agrees with $\theta=0$ being the ground state.

From Eqs. (1), (11), and the boundary conditions $\dot{\theta}(0,t)=\omega_R$, $\dot{\theta}(\infty,t)=0$, we obtain the conservation law

$$\frac{\partial E(t)}{\partial t}=\tfrac{1}{2}\omega_R^2\ , \qquad (12)$$

where $\omega_R=\mu\mathcal{E}_0/\hbar$.

Invariance of (10) under translations in space leads to the momentum

$$P(t)=\int_0^\infty dt\,\theta'(\dot{\theta}+\tfrac{1}{2}c\theta')\ . \qquad (13)$$

From Eqs. (1) and (13) follows the conservation law

$$\frac{dP(t)}{dt}=\alpha c(1-\cos\omega_R t)-\tfrac{1}{2}\omega_R^2\ . \qquad (14)$$

Note that the energy is increasing in a constant rate [Eq. (12)], while the momentum changes in an oscillating fashion [Eq. (14)].

In addition to conservation laws there is another important restriction on the solution which is an asymptotic area theorem. The usual area theorem[1-3] does not hold in our case since it assumes that the pulses separate and $\mathcal{E}(z,\infty)\simeq 0$. We show here that the analogous area theorem in our case is the equation

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{E}(z,t')dt'=\mathcal{E}_0\ . \qquad (15)$$

This means that the input area $\mathcal{E}_0 t$ equals the area $\int_0^t \mathcal{E}(z,t')dt'$ at any z asymptotically, i.e., only a finite difference may exist when $t\to\infty$.

The proof is a simple consequence of the continuity of the function $\theta(z,t)$. The difference $\theta(z,t)-\theta(0,t)$ must be finite for any finite z, even in the limit $t\to\infty$; otherwise $\theta(z,\infty)$ is not continuous as function of z. Therefore $[\theta(z,t)-\theta(0,t)]/t\to 0$ as $t\to\infty$, and from the definition Eq. (2) the result Eq. (15) is obtained.

We proceed now to apply these analytic results to the soliton-lattice solutions, and see if they can be a consistent asymptotic solution.

The soliton lattice is the general solution of Eq. (1) which is a function of $t-z/v$. Equation (1) has then the form

$$\ddot{\theta}\left[\frac{c}{v}-1\right]=\alpha c\sin\theta \qquad (16)$$

with solutions given by the Jacobian elliptic function[7] with parameter λ

$$\sin\tfrac{1}{2}(\theta-\pi)=sn\left[\frac{t-z/v}{\tau};\lambda\right] \qquad (17)$$

and the relation

$$\frac{1}{v}-\frac{1}{c}=\frac{\alpha\tau^2}{\lambda^2}\ . \qquad (18)$$

Thus only two of the parameters τ, v, and λ are independent. The angle θ in Eq. (17) increases by 2π when the time is increased by $2\tau K(\lambda)$. As $\lambda\to 0$

this sequence of 2π pulses reduces to well-separated solitons. The electric field is given by

$$\mathcal{E}(z,t)=\frac{2\mu}{\hbar\tau}dn\left[\frac{t-z/v}{\tau};\lambda\right]. \tag{19}$$

For $\lambda\rightarrow 1$ this reduces to the well-known hyperbolic secant pulse.

Consider first the restriction of Eq. (15) on the solution (19), assuming it is the correct asymptotic behavior. The average of Eq. (19) on one period is $(\mu/K)\pi/(\tau K)$ and Eq. (15) yields

$$\tau=\frac{\pi}{\omega_R K(\lambda)}. \tag{20}$$

In terms of the dimensionless variables Eq. (5) the soliton-lattice solution has now the form

$$\epsilon(x,y)=\frac{2K(\lambda)}{\pi}dn\left[\frac{K(\lambda)}{\pi}y-\frac{\pi}{\lambda^2 K(\lambda)}x;\lambda\right] \tag{21}$$

with the single free parameter λ. The periodicity of this solution in the y variable is 2π. This is very close to the result of the numerical analysis. This closeness was also noted by Crisp,[6] and its proof is the asymptotic area theorem Eq. (15).

The numerical periodicity, defined as the distance between nearest maxima, is, however, not exactly 2π; it varies in the range 6.3−6.5. The variations in this periodicity are not a result of numerical errors, since our results have converged to at least six digits; therefore this periodicity is significantly different from 2π. The variations in the periodicity reflect our result that the asymptotic solution is not a singly periodic function, although the *dn* function is a good approximation in some range of x and y.

The solution (21) gives also the soliton-lattice velocity as

$$U=[\lambda K(\lambda)/\pi]^2. \tag{22}$$

From the slope of Fig. 6, $U\simeq 0.85$ which yields $\lambda=0.997$. Thus the pulse train, or at least its front, propagates with an "effective" λ which is very close to one. This is consistent with the tendency of the pulses to break up, i.e., $\lambda\rightarrow 1$.

Finally consider the restrictions imposed by the conservation laws. If all the input energy results in a propagating soliton lattice, then the average energy in one period times the velocity v should equal the input power. The ratio of output to input powers [the two sides of Eq. (12)] is then

$$R_1=\left[\frac{2K}{\pi}\right]^2\left[\frac{E}{K}+\frac{\frac{\alpha c}{\omega_R^2}\left[\frac{\pi}{\lambda K}\right]^2(\lambda^2-1)}{1+\frac{\alpha c}{\omega_R^2}\left[\frac{\pi}{\lambda K}\right]^2}\right], \tag{23}$$

where $K=K(\lambda)$, $E=E(\lambda)$ are the complete elliptic integrals.[7]

The average rate of momentum increase divided by the average rate of input momentum increase [Eqs. (13) and (14)] is given by

$$R_2=\frac{\frac{4}{\pi^2}KE\left[\frac{\alpha c}{\omega_R^2}\left[\frac{\pi}{\lambda K}\right]^2-1\right]}{\left[\frac{2\alpha c}{\omega_R^2}-1\right]}. \tag{24}$$

Note that the right-hand side of Eq. (14) (rate of input momentum) oscillates in time; however, the average on the period of (21) yields $\langle\cos\omega_R t\rangle=0$.

The conservation laws are satisfied on the average if $R_1=1$ and $R_2=1$. The solution of these equations yields the value of λ as function of the parameter $\alpha c/\omega_R^2$ as shown in Fig. 7. Clearly the value of λ depends on $\alpha c/\omega_R^2$ and cannot be described by a universal value. The two curves in Fig. 7 are rather close to each other, indicating that the soliton-lattice description may be a good approximation, although not an exact one.

The conservation laws are an exact result, valid for any time. This of course cannot be achieved just by the solution Eq. (21). The assumption on the average conservation law means that Eq. (23) is not exact for all times, but the energy and momentum input redistribute within each period to agree with Eq. (21). As we have shown, even this as-

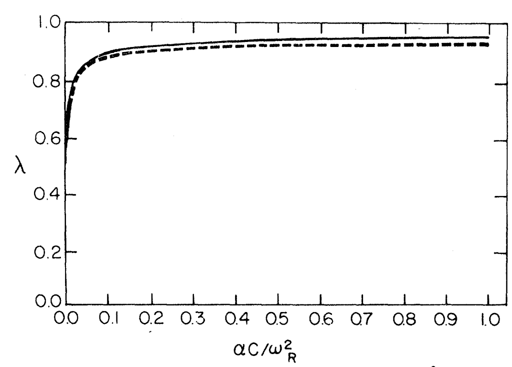

FIG. 7. Values of λ as function of $\alpha c/\omega_R^2$, which satisfy the average energy conservation (full line) and the average momentum conservation (dashed line).

sumption cannot be exact. In addition there are higher-order conservation laws[2,12] which cannot all be simultaneously satisfied for Eq. (21); just as the curves in Fig. 7 are not identical.

Crisp[6] suggested that energy conservation implies $\lambda^2 K/E=1$. This is independent of $\alpha c/\omega_R^2$, unlike our result (23). Conservation laws in the variables x,y can also be constructed, and then they are independent of $\alpha c/\omega_R^2$. However, these conservation laws are not useful since the boundary value $\epsilon(x=\infty,y)$ is not known. The condition $\epsilon(z,t)=0$ for $t<z/c$, which leads to $\epsilon(\infty,t)=0$ for *any t* [and to Eqs. (12) and (14)], is equivalent to $\epsilon(\infty,y)=0$ only for $y<0$ which is not useful.

IV. CONCLUSIONS

We have shown that a single soliton-lattice —type solution cannot be an asymptotic solution of Eq. (1) with the boundary conditions Eqs. (3) and (4). The soliton-lattice functions are however a useful qualitative way of describing the solutions. For $x=1$ we obtain $\lambda\simeq 0.8$, increasing with x to $\lambda\simeq 0.95$ for $x=11$. This shows the tendency of the pulses to separate as they propagate into the medium. The mean separation, however, of the pulses cannot change because of the asymptotic area theorem Eq. (15). Therefore the pulse separation is manifested by the pulses becoming narrower with larger amplitude, corresponding qualitatively to λ closer to 1.

We have considered here the idealized equation of self-induced transparency. The inclusion of inhomogeneous broadening is usually believed not to affect the solution, except for renormalizing some parameters.[1-6] In particular it was shown that the soliton lattice Eq. (17) is a solution of the equations with inhomogeneous broadening.[4,5] The effects of relaxation are less clear[6] and further work on this aspect is needed.

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