Quantum Hall Effects in Layered Disordered Superconductors

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Layered singlet paired superconductors with disorder and broken time reversal symmetry are studied, demonstrating a phase diagram with charge-spin separation in transport. In terms of the average intergrain transmission and the interlayer tunneling we find quantum Hall phases with spin Hall coefficients of $\sigma_{xy}^{\mathrm{spin}} = 0, 2$ separated by a spin metal phase. We identify a spin metal-insulator localization exponent as well as a spin conductivity exponent of ≈ 0.96. In the presence of a Zeeman term an additional $\sigma_{xy}^{\mathrm{spin}} = 1$ phase appears.

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The problem of quasiparticle transport and localization in disordered superconductors is of considerable interest in view of experimental activity on the high T_c cuprates as well as theoretical realization that disordered superconductors provide new symmetry classes of random matrix theory [1]. Of particular interest is class C for which the Hamiltonian breaks time reversal symmetry but spin rotation invariance remains intact. Physically, it can be realized in materials consisting of singlet superconductor grains in a magnetic field or else, by a superconductor in the absence of magnetic field whose order parameter breaks time reversal invariance, such as $d+id'$. Class C can therefore be realized by high T_c compounds where d-wave pairing is well established. In fact, $d + id'$ pairing has been suggested [2], in particular, in overdoped compounds or as field induced [3].

Transport properties of random superconductors are unusual since a quasiparticle does not carry charge, being screened by the condensate, while the singlet paired condensate does not transport spin. Furthermore, the gapless nature of d-wave pairing with low lying quasiparticle excitations leads to a rich phase diagram in two dimensions (2D) with spin quantum Hall phases [4,5], spin insulators, and spin metals [6,7]; a metallic phase was also found for triplet pairing [8].

The usual quantum Hall system in 2D as well as its extension to three-dimensional (3D) layered system have been studied by a network model [9,10] which consists of a lattice of nodes connected by links. In 2D, the unidirectional propagation on links is described by random phases, corresponding to a group U(1), while transfer at nodes is controlled by a parameter which determines the critical point. The transfer matrix of the network model can be efficiently evaluated identifying the critical behavior; e.g., the localization exponent is $\nu_{\mathrm{QH}} \approx 2.5$. The 2D class C problem has recently been studied by a network model [4] where propagation on links of particle-hole spinors via the Bogoliubov–de Gennes Hamiltonian is realized by random SU(2) matrices. The quantized spin Hall conductance σ_{xy} is shown to jump by two units at a critical point of a new universality class with a localization exponent $\nu_{2\mathrm{D}} \approx 1.12$; an exact mapping on a classical percolation problem [11,12] has found $\nu_{2\mathrm{D}} = 4/3$. The spin rotation invariance can be broken by having a different transmission for particles and holes [4], e.g., a Zeeman term. The phase diagram has then three phases with $\sigma_{xy} = 0, 1, 2$ and a localization exponent ν_{QH} of the usual U(1) theory.

Recent advances in measuring spin transport [13] allow probing of these unusual predictions, provided that deviations from the critical point can be controlled. In the usual quantum Hall effect, this is achieved by the position of the Fermi energy relative to that of an extended state in a Landau band. In a superconductor, the particle-hole symmetry fixes the Fermi energy at the middle of the gap, and the relative position of states is not directly tuned by the overall density. It was in fact suggested that changing the strength of disorder can lead to quantum Hall transitions [5], at least for weak breaking of time reversal symmetry.

An important insight into the nature of $d + id'$ superconductors comes from studying their edge states [2,3,14] which provide a realization of our network model and identify its parameters. In the d-wave case, a prominent zero bias anomaly [2,3] has identified a surface state at zero energy. The $d + id'$ case allows current carrying chiral states that split the zero bias anomaly as seen in the overdoped compounds [3]. The chirality of these edge states leads directly to quantized Hall conductance [5,14]. In contrast, charge transport of superconducting grains is dominated by the randomness of the Josephson coupling between the grains; the phase correlation between grains is lost in 2D at a critical value of disorder, as shown in an XY model with random phase shifts [15].

In the present work we solve a network model for a layered 3D random superconductor. We find three phases which are identified as spin insulators with Hall coefficient 0 and 2, respectively, and a spin metal phase. We also identify the localization exponent at the insulator-metal transition, $\nu_{3\mathrm{D}} \approx 0.96$. When the spin rotation symmetry is broken by a Zeeman term, an additional phase with Hall coefficient $\sigma_{xy}^{\mathrm{spin}} = 1$ is found. The spin

metal phase is a realization of a proposed phase with nonzero spin diffusion constant at zero temperature [6]. We identify the conductivity of the spin metal and find its critical exponent to be ν_{3D} as well. Finally, we identify the physical parameter that controls criticality of spin transport, namely, the average transmission of quasiparticles between grains. This then demonstrates spin-charge separation in the sense that their critical behavior relevant to transport is controlled by distinct parameters.

The 3D network model consists of layers of 2D lattices. Each 2D lattice has nodes, which are connected by links [9]. Quasiparticles propagate unidirectionally on links, hence their transfer matrix is equivalent to the evolution operator, which for a singlet superconductor is a SU(2) matrix describing an Andreev process where particle and hole components mix [4],

$$\mathbf{T}_1 = \begin{pmatrix} e^{i\delta_1}\sqrt{1-x} & -e^{i\delta_2}\sqrt{x} \\ e^{-i\delta_2}\sqrt{x} & e^{-i\delta_1}\sqrt{1-x} \end{pmatrix}, \quad (1)$$

where δ_1, δ_2, x are random, $0 \le \delta_1$, $\delta_2 < 2\pi$, $0 \le x < 1$.

The propagation between grains, i.e., at nodes of the network, is determined as follows. At each node we have two incoming and two outgoing links with particles and holes separately scattered. The transmission probability at a node is parameterized in the form $T_0 = [1 + \exp(-\pi\epsilon)]^{-1}$, so that for $\epsilon = 0$ the transmission equals reflection, i.e., maximal mixing of all links. In the following, we allow also a Zeeman parameter Δ which breaks spin rotation invariance. The transfer matrix across a node is then

$$\mathbf{T}_2 = \begin{pmatrix} \sqrt{1 + e^{-\pi(\epsilon \pm \Delta/2)}} & e^{-\pi(\epsilon/2 \pm \Delta/4)} \\ e^{-\pi(\epsilon/2 \pm \Delta/4)} & \sqrt{1 + e^{-\pi(\epsilon \pm \Delta/2)}} \end{pmatrix}, \quad (2)$$

where the $\pm$ sign corresponds to particle with spin-up or hole with spin-down, respectively.

The final ingredient of the 3D network model consists of additional nodes [10] connecting neighboring layers. Placing 2D layers one on top of the other allows for links belonging to neighboring layers to form nodes and quasiparticles scatter. The matrix describing this scattering is

$$\mathbf{T}_3 = \begin{pmatrix} \sqrt{1-t^2} & t \\ -t & \sqrt{1-t^2} \end{pmatrix}. \quad (3)$$

Consider a system of size $M \times M \times L$ where M is the number of links in one layer (with two channels per link) and $L \to \infty$ (typically $L \approx 10^5$) is its length. For a given M, the eigenvalues of $\ln(T^\dagger T)$, where T is the full transfer matrix, behave as $-2\lambda_n L$, defining the Lyapunov exponents $\{\lambda_n\}$; the smallest positive one, λ_1, defines the localization length $\xi_M = 1/\lambda_1$. The M dependence of ξ_M/M identifies the phases: (i) a decreasing ratio corresponds to localized state, i.e., a spin insulator, (ii) a constant ratio corresponds to a critical state, and (iii) an increasing ratio corresponds to a spin metal. The phase diagram for the class C network model (with $\Delta = 0$) is displayed in Fig. 1. Square boxes represent critical $\epsilon_{cr}(t)$ lines. The particle-hole symmetry of the superconductor ensures a degeneracy at the critical point $\epsilon = t = 0$, i.e., the Hall coefficient changes by two units. Furthermore, in the clean limit, the Hall conductance has two units [14]; this corresponds to transmission $T_0 = 1$, i.e., $\epsilon > 0$ and large. Hence there are three distinct phases: Hall insulator with Hall conductance $\sigma_{xy}^{\text{spin}} = 1$, spin metallic phase, and a quantized spin Hall phase with $\sigma_{xy}^{\text{spin}} = 1$.

The width $W(t)$ in ϵ of the metallic region increases with t, and is expected to behave as [10] $W(t) \sim t^{1/\nu_{2D}}$, where ν_{2D} is the localization length exponent in 2D. To show this, consider a 2D isolated layer with a mean level spacing $\sim 1/\xi_{2D}^2$ (ξ_{2D} the 2D localization length) and with 2D wave functions having a random phase correlated on a scale ℓ. The overlap of neighboring layers has then $(\ell/\xi_{2D})^2$ terms with random phases so that their sum yields an interlayer coupling $\sim t/\xi_{2D}$. At the mobility edge the mean level spacing is of the same order as the interlayer coupling according to the Thouless criterion, hence $\epsilon \sim t^{1/\nu_{2D}}$. The curve in Fig. 1. represents the least square fit for the data, producing $W \sim t^{1/1.2}$, which is in good agreement with our previous value [4] of ν_{2D}.

The divergence of the localization length at $\epsilon_{cr}(t)$ identifies the localization exponent of a spin insulator-metal transition in 3D. We have evaluated the critical exponent at the symmetric point $t = 1/\sqrt{2}$ (larger t maps into a smaller t by rearranging layer indices) and found $\nu_{3D} = 0.96$ fitting both the insulator and metal sides, as shown in Fig. 2 with an estimated error of ± 0.1; a similar value of ν_{3D} is found for $t = 0.1$.

Figure 2 shows that in the metallic phase, ξ_M/M increases approximately linearly with M. It was proposed [16] that this identifies the 3D conductivity as $\sigma_{xx} \sim (\epsilon - \epsilon_{cr})^{\nu_{3D}}$. This derivation needs to be revised since in the 3D limit, the conductivity involves many Lyapunov exponents λ_n. The multichannel conductance is given by [17,18] $g = \sum_n [1 + \cosh(\lambda_n L)]^{-1}$. For a few channels $M^2 \ll L$, the lowest Lyapunov dominates, but in the 3D limit $M^2 \gg L$, the number of terms N_{eff} that contribute to

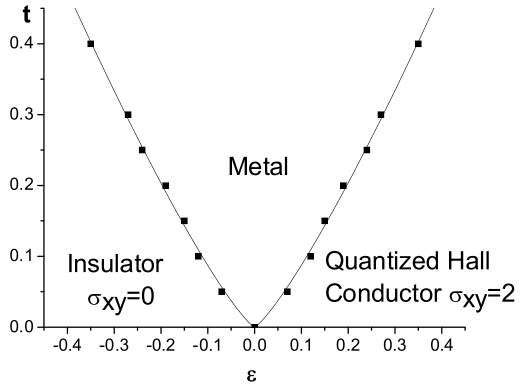

FIG. 1. Phase diagram of the 3D network model with spin-rotational symmetry. The solid lines describe the fit $t \sim |\epsilon|^{1.2}$.

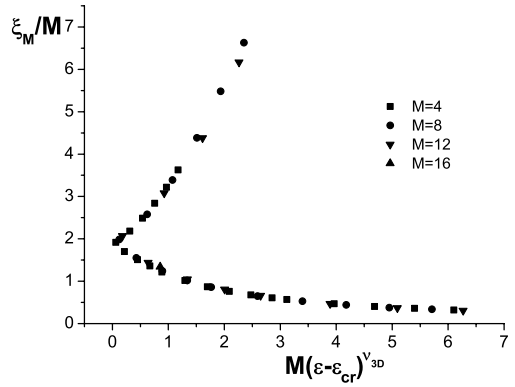

FIG. 2. Scaling of renormalized localization length for the 3D network model with spin-rotational symmetry at $t = 1/\sqrt{2}$ showing an exponent $\nu_{3D} = 0.96$. The lower and upper branches correspond to the insulating $\epsilon > \epsilon_{cr}$ and metallic $\epsilon < \epsilon_{cr}$ phases, respectively.

g is large. In fact, for many channels, the rigidity in the spectrum of $T^\dagger T$ suppresses fluctuations in $\{\lambda_n\}$ and one expects [18] $\lambda_n = n\lambda_1$. Hence $g \approx N_{\rm eff} \approx 1/(\lambda_1 L)$. The conductance has then the form

$$g \approx N_{\rm eff} \approx \frac{M}{L}[a(\epsilon - \epsilon_{\rm cr})^{\nu_{3D}} M + b]. \tag{4}$$

This shows that the conductivity in 3D is indeed $\sigma_{xx} \equiv gL/M^2 \sim (\epsilon - \epsilon_{\rm cr})^{\nu_{3D}}$. On the critical line $\epsilon = \epsilon_{\rm cr}$, the conductance $g = Mb/L$ is limited to the surface area.

Consider next the $\Delta \neq 0$ case with broken spin rotation symmetry. At $\Delta = 2$, e.g., the 2D system ($t = 0$) is critical at $\epsilon_{\rm cr} = \pm 0.64$ with a critical exponent $\nu_{\rm QH} \approx 2.5$ of the usual quantum Hall system. At $t \neq 0$, we expect each of the critical states to split into two with a band of metallic states between them (as for $\Delta = 0$); however, it is not obvious whether the two internal critical curves merge or do not cross as t increases. We find merging of these lines, producing a four-phase diagram as shown in Fig. 3. Both outer critical lines scale as $t \sim [|\epsilon| - 0.64]^{\nu_{\rm QH}}$

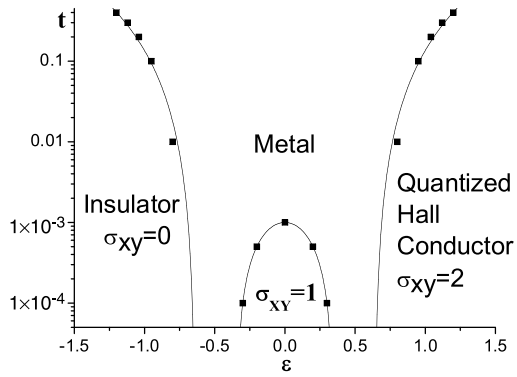

FIG. 3. Phase diagram of the 3D network model with $\Delta = 2$ showing an additional phase with $\sigma_{xy}^{\rm spin} = 1$. The outer solid lines describe the fit $t \sim [|\epsilon| - 0.64]^{2.5}$, while the inner solid line is a guide to the eye.

in agreement with the argument above. The inner curve is affected by both critical points and therefore deviates from this scaling form.

Figure 3 shows the existence of a new phase with $\sigma_{xy}^{\rm spin} = 1$, which becomes metallic at very low values of t, e.g., $t = 0.001$ at $\epsilon = 0$. We note that at $t = 0$, a single spin state becomes extended at $\epsilon_{\rm cr}$, while the other spin state becomes extended independently at $-\epsilon_{\rm cr}$. For $t < 0.001$, these extended states produce two metallic bands which do not overlap, hence when the chemical potential is in between these bands, the $\sigma_{xy}^{\rm spin} = 1$ phase emerges. When $t > 0.001$, these bands overlap and a $\sigma_{xy}^{\rm spin} = 1$ phase is not possible. We emphasize that for $t \neq 0$ there is a single simply connected metallic phase, hence in this phase the two extended spin states mix via interlayer coupling, unlike the situation at $t = 0$.

We proceed to evaluate the localization length exponent for $\Delta = 2$ and different t. For $t = 1/\sqrt{2}$ [maximal mixing of Eq. (3)], we find $\nu = 0.85$ while for $t = 0.1$ we find $\nu \approx 0.93$, fairly close to ν_{3D}. It differs significantly from the value 1.45 found for the 3D U(1) system [10]. This is consistent with our finding of a single metallic phase in which extended states with both spins are mixed. Thus at $t = 0$ (extended states at $\pm\epsilon_{\rm cr}$ are decoupled), the symmetry is reduced to U(1) producing the $\nu_{\rm QH}$ exponent, while the $t \neq 0$ metallic phase (mixed extended states) seems to have the same symmetry for $\Delta = 0$ or $\Delta \neq 0$.

Based on this analysis, we propose a three-parameter scaling function near the multicritical point $\epsilon = \Delta = t = 0$

$$\frac{\xi_M}{M} = f(\epsilon^{\nu_{2D}} M, \Delta^{\mu} M, t^{\alpha} M). \tag{5}$$

On the critical surface, the scaling function is M independent, hence for $\Delta = 0$ the critical line is $\epsilon_{\rm cr} \sim t^{\alpha/\nu_{2D}}$. As discussed above, $W(t) \sim \epsilon_{\rm cr} \sim t^{1/\nu_{2D}}$; therefore, $\alpha = 1$. We demonstrate in Fig. 4 scaling along the line $\epsilon = \Delta = 0$ with $\alpha = 1$. Note that when t is too large, approaching the symmetric point $1/\sqrt{2}$, scaling is not expected. The

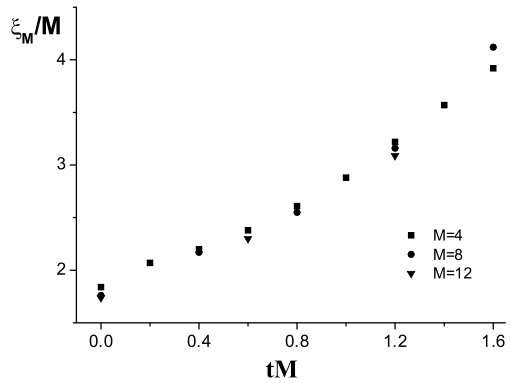

FIG. 4. Scaling of renormalized localization length for $\epsilon = \Delta = 0$ as function of interlayer coupling.

analysis above yields $\sigma_{xx} \sim t$ when approaching the multicritical point.

Finally we consider a realization of the parameter which drives the phase transitions. Correspondence with edge states of the $d + id'$ system shows that the transition is driven by the average value T_0 of quasiparticle transmission between grains, which determines ϵ via $T_0 = [1 + \exp(-\pi\epsilon)]^{-1}$. Following the suggestion that, at least for weakly broken time reversal symmetry, disorder may drive the transitions [5], we have performed further simulations of the 2D network searching for the effect of randomness in ϵ. We have found this randomness to have a negative scaling exponent, i.e., an irrelevant variable. In fact, the mapping to a percolation problem [12] leads to a distribution in the percolation parameter which is indeed irrelevant. Hence, in our system where time reversal symmetry breaking is fully developed, disorder does not drive a spin quantum Hall transition. In contrast, charge transport is dominated by the randomness of the Josephson coupling between superconducting grains; the phase correlation between grains is lost in 2D at a critical value of disorder as shown in an XY model with random phase shifts [15]. We conclude then that a superconductor-insulator transition for charge transport is disconnected from that of quasiparticle spin transport. Therefore, although the quasiparticles are repeatedly scattered by the condensate random order parameter, spin-charge separation in transport persists on long scales.

As a concrete realization, we consider two grains of $d + id'$ superconductors with parallel edge states along an axis x. An impurity provides an intergrain scattering potential $Va\delta(x)$, where a is a lattice constant. The right and left moving edge state $\psi_R(x)$ and $\psi_L(x)$ then satisfy

$$\begin{aligned} -iv\partial_x\psi_R(x) + Va\delta(x)\psi_L(x) &= E\psi_R(x) \\ iv\partial_x\psi_L(x) + Va\delta(x)\psi_R(x) &= E\psi_L(x), \end{aligned} \tag{6}$$

where E is an energy eigenvalue. Here v is the edge state velocity, $v \approx a\Delta_{xy}$ for a (110) surface and $v \approx a\Delta_{x^2-y^2}$ for a (100) surface [14] where Δ_{xy} and $\Delta_{x^2-y^2}$ are the gaps of $d_{x^2-y^2}$ and d_{xy}, respectively, with $\Delta_{xy} \ll \Delta_{x^2-y^2}$.

The transmission from an incoming $\psi_R(x)$ to an outgoing $\psi_L(x)$ is readily evaluated as

$$T_0 = 4(Va/2v)^2/[1 + (Va/2v)^2]^2. \tag{7}$$

Note that T_0 has a maximum of 1 at $Va/2v = 1$ and decreases at large V. The transmission needed for exhibiting an extended state, $T_0 = 1/2$, is achieved at $V \approx \Delta'$ for (110) edges or $V \approx \Delta$ for (100) edges, i.e., a much weaker coupling in the former case.

In conclusion, we have demonstrated spin-charge separation in transport: spin transport and related QH transitions are controlled by the average intergrain transmission while charge transport and superconducting correlation is controlled by the amount of disorder in the intergrain (Josephson) coupling. We show that interlayer coupling leads to a new spin metal phase and identify the localization exponent for the spin insulator-metal transition as $\nu_{3D} \approx 0.96$. The latter is also the spin conductivity exponent when approaching the transition from the metallic side.

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