Brief Review

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VARIOUS FACETS OF CHALKER–CODDINGTON NETWORK MODEL

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We briefly discuss various applications of the Chalker–Coddington network model, starting with the original one, proposed to describe inter-plateaux transition in the integer quantum Hall effect (IQHE). Next, we present generalization appropriate for the IQHE allowing to include spin, and conclude with recent applications to dirty superconductors. We then describe how numerical calculations on an open network produce data for the localization length behavior in the metal-insulator transition, whereas calculations on the closed system allow elucidation of various levels statistics. We also discuss how numerical algorithm for systems with additional symmetries is modified in order to improve the accuracy. Finally, results for the nearest-neighbor spacing distribution in dirty superconductors are presented.

Keywords: Network model; quantum Hall effect; new symmetry classes.

1. Introduction

Numerical models are usually constructed for describing particular phenomena and for calculating some specific properties. They serve that purpose, produce some results, and usually have no future. In this brief review, we discuss a model which was initially designed for the critical behavior at the integer quantum Hall plateau transition, and in this sense could have followed the standard route of numerical models. In fact, this model turned out to allow generalizations, describe more involved phenomena, make analytic predictions, produce results for different physical quantities, and in general, be applicable to systems which are very different from the original quantum Hall system.

2. One-Channel Network Model

The original network model[1] was proposed to describe transitions between plateaux in the quantum Hall effect (QHE). QHE is realized in a two-dimensional electron gas subject to a strong perpendicular magnetic field and a random potential. When random potential varies smoothly (its correlation length is much larger than the mag-

netic length), a semi-classical description becomes relevant: electrons move along the lines of constant potential. When two equipotential lines come close to each other (near a saddle point), tunneling is feasible. In the network model, electrons move along unidirectional links forming closed loops in analogy with semi-classical motion on contours of constant potential. Scattering between links is allowed at nodes in order to encode tunneling through saddle point of the potential landscape. Propagation along links yields a random phase ϕ. Thus, links are presented by diagonal matrices with elements in the form $\exp(i\phi)$. Transfer matrix for one node relates a pair of incoming and outgoing amplitudes on the left to a corresponding pair on the right; it has the form

$$\mathbf{T} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}. \tag{1}$$

The node parameter θ is related to the electron energy in the following way

$$\epsilon = -\frac{2}{\pi}\ln(\sinh\theta)\,, \tag{2}$$

where ϵ is a relative distance between the electron energy and the barrier height. It is easy to see that the most "quantum" case (equal probabilities to scatter to the left and to the right) is at $\epsilon = 0$ ($\theta = 0.8814$); numerical calculations show that there is an extended state at that energy.

Numerical simulations on the network model are performed in the following way: one study system with fixed width M and periodic boundary conditions in the transverse direction. Multiplying transfer matrices for N slices and then diagonalizing the resulting total transfer matrix, it is possible to extract the smallest Lyapunov exponent λ (the eigenvalues of the transfer matrix are $\exp(\lambda N)$). The localization length ξ_M is proportional to $1/\lambda$. Repeating calculations for different system widths and different energies, it is possible to show that the localization length ξ_M satisfies a scaling relation

$$\frac{\xi_M}{M} = f\left(\frac{M}{\xi(\epsilon)}\right). \tag{3}$$

In the QHE, the thermodynamic localization length $\xi(\epsilon) \sim |\epsilon|^{-\nu}$ and $\nu = 2.5 \pm 0.5$. This is the main result[1] and it is in a good agreement with experimental data for spin-split resolved levels,[2] numerical simulations using other models[3] and semi-classical argument[4,5] that predicts $\nu = 7/3$.

3. Two-Channel Network Model

A natural generalization of the network model includes spin or mixing of the two lowest Landau levels. It is achieved by allowing each link to carry two channels — both in the same direction. Two states can mix on the link but scatter separately at the node. Energies of both states are equal but are differently distributed between guiding center motion and Landau (spin) level allowing, therefore, two lowest Landau levels or spin-up and spin-down levels. A certain parameter

Δ describes the relative energy distance between the two levels and effectively measures the magnetic field. The mixing matrix on the link is a 2×2 unitary matrix

$$\mathbf{U} = e^{i\delta} \begin{pmatrix} e^{i\alpha} \cos\phi & e^{i\gamma} \sin\phi \\ -e^{-i\gamma} \sin\phi & e^{-i\alpha} \cos\phi \end{pmatrix}. \tag{4}$$

Results for spin degenerate levels show that there are two separate critical energies.[6] Performing numerical calculations for different energies and Δ, we trace[7] the dependence of the extended states energies on magnetic field without fitting parameter and confirm the conjecture on the levitation of extended states in low magnetic fields.[8]

The symmetries of the transfer matrix on the link reflect the symmetries of the underlying Hamiltonain and can therefore describe different physical systems. We have already seen how a group symmetry of the link matrix corresponds to the lowest Landau level (U(1) symmetry) and to two spins or Landau levels (U(2) symmetry). Two other symmetries SO(2) and U(1) $\times$ SO(2) exhibit the same behavior as a single channel model. This is not surprising due to the commutativity of the U matrices on the links.

4. New Symmetry Classes

It turns out that the "nearest neighbors" of the first two models, namely, systems with O(1) and SU(2) symmetries are of great interest and correspond to very nontrivial physical systems. Altland and Zirnbauer[9] considered properties of quasiparticles in disordered superconductors that are governed by a quadratic Hamiltonian, which may include effects of disorder in both the normal part and the superconducting gap function. Such Hamiltonians are representatives of a set of symmetry classes different from the three classes which are familiar both in normal disordered conductors and in the Wigner–Dyson random matrix ensembles. A list of additional random matrix ensembles, determined by these new symmetry classes, has been established. These additional random matrix ensembles describe zero-dimensional problems, and are appropriate to model a small grain of a superconductor in the ergodic highly conducting limit. In our work we have extended the study of classes C and D into two dimensions and found transitions between metallic, localized, or quantized Hall phases for quasiparticles.

4.1. *Class C*

Hamiltonian with broken time reversal symmetry and spin rotational symmetry intact belongs to class C in Altland and Zirnbauer notation. It can be realized when grains of a single superconductor surrounded by normal metal are subject to a perpendicular magnetic field. The associated change in quasiparticle dynamics must be probed by spin transport, rather than charge transport, since quasiparticle charge density is not conserved. This symmetry class when mapped onto CC

network model corresponds to SU(2) matrices on the link. Scattering at the nodes is parameterized by $\epsilon \pm \frac{1}{2}\Delta$. The value of ϵ determines the Hall conductance of the system, as measured at short distances: varying ϵ drives the model through the delocalization transition (in this sense ϵ corresponds to a Fermi level as in the standard CC model). A non-zero value for Δ breaks spin-rotation invariance, and will in fact change the universality class for the transition. The results[10] show that the renormalized localization length ξ_M/M_l (M_l is a system width) is described near the fixed point ($\epsilon = \Delta = 0$) by a two parameter scaling function

$$\xi_M/M_l = f(\epsilon M_l^{1/\nu}, \Delta M_l^{1/\mu}), \tag{5}$$

with $\nu = 1.12$ and $\mu \approx 1.45$. It was also shown that as Δ approaches zero, extended states coalesce, having a separation, $2\epsilon_c \propto \Delta^{\mu/\nu}$ which is much smaller than Δ, their separation in the absence of coupling between the two spin orientations. Finally, in contrast with the conventional plateau transition, the Hall conductance for conserved quasiparticles ($\Delta = 0$) changes at this transition by two units.

4.2. *Class D*

The last symmetry class treated by CC model is class D. The symmetry may be realized in superconductors with broken time-reversal invariance, and either broken spin-rotation invariance (as in *d*-wave superconductors with spin-orbit scattering) or spinless or spin-polarized fermions (as in certain *p*-wave states). The associated changes in quasiparticle dynamics must be probed by energy transport, since neither charge density nor spin are conserved. A Bogoliubov–de Gennes Hamiltonian with this symmetry may be written in terms of a Hermitian matrix.[9] The corresponding time evolution operator is real, restricting the generalized phase factors to be O(N) matrices for a model in which N-component fermions propagate on links, and to the values ± 1 for $N = 1$, the case that was studied. We define two models: uncorrelated O(1) model, where phases on the links are independent random variables, and a model first introduced by Cho and Fisher (CF)[11] where scattering phases with the value π appear in correlated pairs (see details below). Each model has two parameters. The first one is a disorder concentration W, such that there is a probability $W(1 - W)$ to have phase $0(\pi)$ on a given link. The second parameter is an energy ϵ describing scattering at the nodes. We have found[12] that in the uncorrelated O(1) model, all states are extended independent of ϵ and W. For the CF model, the phase diagram in the ϵ–W plane has three distinctive phases: metallic, and two insulating phases characterized by different Hall conductivities. The sensitivity to the disorder is a distinctive feature of class D.

The existence of region of extended states means that the smallest Lyapunov exponent at each particular energy is zero or extremely small. We wish to discuss this point in some detail in order to demonstrate the power of the CC model. First, we present an analytic argument for this result, and then show how to modify the numerical algorithm employing additional symmetries of the system that increases accuracy of calculations.

Consider the uncorrelated O(1) model with $M = 2$. It has two eigenvectors $(1, -1)^T$ and $(1, 1)^T$. The effect of one node and one link transfer matrix on the first eigenvector is

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \exp(-A\theta) \begin{pmatrix} 1 \\ -A \end{pmatrix} \tag{6}$$

and after many iterations

$$\cdots = \exp[(-A - AB - ABC - \cdots)\theta] \begin{pmatrix} 1 \\ -ABC \cdots \end{pmatrix}, \tag{7}$$

where $A, B, C, \ldots$ assume values $+1$ with probability W, and -1 with probability $1-W$. The same procedure with the eigenvector $(1, 1)^T$ produces the same result as Eq. (7), with all signs reversed. The weighted averaged value of the exponent (taking into account contributions of both eigenvectors) defines the Lyapunov exponent. We therefore need to find the relative weights of the two eigenvectors. The ergodicity of the system implies that after many iterations, the expression $ABC\cdots$ equals $+1$ (-1) with probability α $(1-\alpha)$ for some constant α. Assuming the same probability after the next step, we find $\alpha W + (1-\alpha)(1-W) = \alpha$, which immediately gives $\alpha = 1/2$, thus both eigenvectors have the same relative weight, and their contributions to the Laypunov exponent cancel each other exactly. We therefore conclude that for $M = 2$, the Lyapunov exponent is exactly zero independent of $\theta(\epsilon)$ and W.

The decomposition of the transfer matrix for the CF model gives

$$\begin{pmatrix} \cosh A\theta & \sinh A\theta \\ \sinh A\theta & \cosh A\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}. \tag{8}$$

The same exercise for the CF model produces $\exp(A\theta)$ for both eigenvectors, the Lyapunov exponent then is zero only when $\langle A \rangle = 0$, i.e. for $W = 1/2$.

The standard method for numerically calculating Lyapunov exponents involves application of transfer matrix for successive slices of the system on a set of M orthogonal vectors, and imposing orthogonality by means of Gram–Schmidt procedure.[13] If all Lyapunov exponents are separated by gaps, this set of vectors converges to the eigenvectors of $T^T T$ associated with the first M exponents (the width of the system is $2M$). Convergence rates are determined by the sizes of gaps between adjacent exponents. In the present case, convergence rates are seriously reduced if the smallest positive Lyapunov exponent ν_1 approaches zero, so that the gap between the smallest exponents vanishes. Moreover, numerical noise ultimately limits the extent of convergence, and leads to an erroneously large value for ν_1. To overcome this flow, the following modification of the numerical algorithm was proposed.[12,14] Consider the transfer matrix T of the whole system in more detail. It has the polar decomposition

$$T = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} A_3^T & 0 \\ 0 & A_4^T \end{pmatrix}, \tag{9}$$

where $A_1, \ldots, A_4$ are $M \times M$ real orthogonal matrices and γ is an $M \times M$ real diagonal matrix. It follows that $T^T T$ is diagonalized by the transformation $B^T T^T T B$, where

$$B = \begin{pmatrix} A_3 & A_3 \\ A_4 & -A_4 \end{pmatrix}. \tag{10}$$

We then impose on the M pertinent vectors (beyond simple orthogonality) an additional constraint that their first M components separately form an orthogonal matrix A_3, and their last M components form another orthogonal matrix A_4, as is evident from Eq. (9). This procedure drastically improves convergence and accuracy of the calculations, confirming analytical arguments presented above for both systems.

5. Level Statistics

Finally, we briefly describe the use of the CC model for level statistics calculations. If, instead of studying transport properties of the system, it is closed up as a torus, then the eigenvalue problem can be addressed. Such a system can be specified using a discrete-time evolution operator, $U(\epsilon)$. For a square network of $\mathcal{N} \times \mathcal{N}$ nodes, U is a $(2 \times \mathcal{N}^2) \times (2 \times \mathcal{N}^2)$ unitary matrix. The action of U on a vector Ψ of flux amplitudes, defined on the start of each link, maps the system on itself, providing therefore an eigenvalue equation $U(\epsilon)\Psi = \Psi$. Unfortunately, it is not possible at present to find numerical solutions of that equation. Instead, Klesse and Metzler[15] proposed to solve the unitary evolution equation

$$U\Psi_n = \exp[i\omega_n(\epsilon)]\Psi_n \tag{11}$$

and to study statistics of ω_n for a given ϵ. The idea behind that proposal is based on two suggestions. First, there are many states even in the narrow window near particular energy ϵ to provide good statistics. Second, the behavior of curves $\omega_n(\epsilon)$ is smooth enough, therefore statistics of ω_n for a given ϵ is expected to be the same as statistics of ϵ_n for $\omega = 0$ (true energy eigenvalues). We have reasons to believe that the second suggestion is justified only after proper unfolding procedure, and have shown[16,17] that for the CF model at $\epsilon = 0$ and $W = 0.1$ (where transition takes place) the nearest neighbor spacing distribution (NNSD) $P(s)$ after proper unfolding procedure is in excellent agreement with the Wigner surmise for the GUE. At $\epsilon = 1$ at the same $W = 0.1$ which, according to our phase diagram,[12] is a localized state, we have found a clear transition to Poissonian statistics.

6. Conclusion

We have briefly described various applications of the CC model for different physical systems. The model allows some powerful analytical arguments as well as numerical simulations of different physical quantities. We have to mention that there are numerous applications of the CC model which are not included in this work such as

formulation in terms of supersymmetry,[18] calculations of mesoscopic fluctuations of Hall conductance[19] and generalization to three-dimensional systems.[20]

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References

1. J. T. Chalker and P. D. Coddington, *J. Phys.* **C21** (1988) 2665.
2. S. Koch, R. J. Haug, K. v. Klitzing and K. Ploog, *Phys. Rev. Lett.* **67** (1991) 883.
3. B. Huckestein and B. Kramer, *Phys. Rev. Lett.* **64** (1990) 1437.
4. G. V. Milnikov and I. M. Sokolov, *JETP Lett.* **48** (1988) 536.
5. V. Kagalovsky, B. Horovitz and Y. Avishai, *Europhys. Lett.* **31** (1995) 425.
6. D. K. K. Lee and J. T. Chalker, *Phys. Rev. Lett.* **72** (1994) 1510; D. K. K. Lee, J. T. Chalker and D. Y. K. Ko, *Phys. Rev.* **B50** (1994) 5272.
7. V. Kagalovsky, B. Horovitz and Y. Avishai, *Phys. Rev.* **B52** (1995) R17044.
8. D. E. Khmelnitskii, *Phys. Lett.* **106** (1984) 182; *JETP Lett.* **38** (1983) 556; R. B. Laughlin, *Phys. Rev. Lett.* **52** (1984) 2304.
9. A. Altland and M. R. Zirnbauer, *Phys. Rev.* **B55** (1997) 1142.
10. V. Kagalovsky, B. Horovitz, Y. Avishai and J. T. Chalker, *Phys. Rev. Lett.* **82** (1999) 3516.
11. S. Cho and M. P. A. Fisher, *Phys. Rev.* **B55** (1997) 1025.
12. J. T. Chalker, N. Read, V. Kagalovsky, B. Horovitz, Y. Avishai and A. W. W. Ludwig, *Phys. Rev.* **B65** (2002) 012506.
13. A. MacKinnon and B. Kramer, *Phys. Rev. Lett.* **47** (1981) 1546; *Z. Phys.* **B53** (1983) 1; J. L. Pichard and G. Sarma, *J. Phys.* **C17** (1981) 411.
14. F. Merz and J. T. Chalker, *Phys. Rev.* **B65** (2002) 054425.
15. R. Klesse and M. Metzler, *Europhys. Lett.* **32** (1995) 229; *Phys. Rev. Lett.* **79** (1997) 721–724.
16. V. Kagalovsky, B. Horovitz and Y. Avishai, *Physica* **E12** (2002) 627.
17. V. Kagalovsky, B. Horovitz and Y. Avishai, *Phys. Status Solidi* **230** (2002).
18. M. R. Zirnbauer, *J. Math. Phys.* **38** (1997) 2007.
19. A. G. Galstyan and M. E. Raikh, *Phys. Rev.* **B56** (1997) 1422; D. P. Arrovas, M. Janssen and B. Shapiro, *ibid.* **B56** (1997) 4741.
20. J. T. Chalker and A. Dohmen, *Phys. Rev. Lett.* **75** (1995) 4496.