Freezing transitions and the density of states of two-dimensional random Dirac Hamiltonians

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(Received 9 August 2001; published 13 March 2002)

Using an exact mapping to disordered Coulomb gases, we introduce a method to study two-dimensional Dirac fermions with quenched disorder in two dimensions that allows us to treat nonperturbative freezing phenomena. For purely random gauge disorder it is known that the exact zero-energy eigenstate exhibits a freezinglike transition at a threshold value of disorder $\sigma=\sigma_{th}=2$. Here we compute the dynamical exponent z that characterizes the critical behavior of the density of states around zero energy, and find that it also exhibits a phase transition. Specifically, we find that $\rho(E=0+i\epsilon)\sim\epsilon^{2/z-1}$ [and $\rho(E)\sim E^{2/z-1}$] with $z=1+\sigma$ for $\sigma<2$ and $z=\sqrt{8\sigma}-1$ for $\sigma>2$. For a finite system size $L<\epsilon^{-1/z}$ we find large sample to sample fluctuations with a typical $\rho_\epsilon(0)\sim L^{z-2}$. Adding a scalar random potential of small variance δ, as in the corresponding quantum Hall system, yields a finite noncritical $\rho(0)\sim\delta^\alpha$ whose scaling exponent α exhibits two transitions, one at $\sigma_{th}/4$ and the other at σ_{th}. These transitions are shown to be related to the one of a directed polymer on a Cayley tree with random signs (or complex) Boltzmann weights. Some observations are made for the strong disorder regime relevant to describe transport in the quantum Hall system.

DOI: 10.1103/PhysRevB.65.125323 PACS number(s): 71.10.Ca, 05.20.−y, 05.50.+q, 64.60.Ak

I. INTRODUCTION

The critical behavior of the plateau transitions in the integer quantum Hall effect (QHE) remains an appealing theoretical challenge. Despite numerous attempts, a calculable theory remains elusive. An equivalent version of the quantum Hall system that is believed to capture the relevant physics corresponds to two-dimensional Dirac fermions in presence of both a random vector and a random scalar potential.[1] Conventional perturbative methods have failed and it is believed that the problem is described by some nonperturbative strong coupling regime.[1,2] Recent works using conformal field theory[3–7] or nonlinear σ models aim at reaching this regime.[8]

One possible route of attack is to use the boson representation[1,9,10] based on the network model.[11] Indeed, the model can be mapped exactly, via bosonization, onto a random sine-Gordon model or equivalently a Coulomb gas (CG) with a specific type of disorder. Although the calculation of the density of states, via the retarded Green's function, corresponds to considering a single CG layer, the full treatment of the quantum Hall transition (both advanced and retarded Green's function) requires to study two-coupled Coulomb-gas layers and remains highly nontrivial in these variables. On the other hand, there has been recent progress in understanding disordered CG, mainly in the context of random gauge XY models[12–18] and in particular the freezing transitions that occur in these systems. Methods, such as fugacity distribution renormalization group (RG)[16,17,19] as well as variational methods,[18] have been developed that seem to capture some of the nonperturbative features of the strong disorder regimes. It is thus of interest to search what can be learned from these methods and to understand, irrespective of formal technicalities, whether the (glass transition) physics that they describe will be part of the QHE strong disorder physics.

In this paper we mainly focus on the detailed understanding of the single-layer problem in the Coulomb-gas formulation with the practical aim of computing the density of states. We also extend our method to the full QH problem, proposing a approach on this venerable problem.

We start by further restricting to the purely random vector-potential disorder model, the scalar random potential will be added later on. This simpler model has been intensively studied[1,7,6,20,21] and is believed to be critical, with a line of fixed points, and a continuously varying dynamical exponent $z(\sigma)$ as a function of random vector-potential disorder strength σ. Some precise results exist for an exactly known zero-energy eigenstate that has the form $\psi(x)=e^{U(x)/2}\psi_0$ where $U(x)/2$ is the primitive of the vector potential. It was found[22] that averaged moments scale with system size L as $\Sigma_x|\psi(x)|^{2q}\sim L^{-\tau(q)}$, such that above a threshold value $\sigma=\sigma_{th}$ of disorder $\tau(q)=0$ for sufficiently large q indicating some kind of localized behavior. Further studies[17] confirmed the existence of a transition at $\sigma=\sigma_{th}$ in the (Gibbs-like) probability measure $|\psi(x)|^2=e^{U(x)}$ (equivalently a freezing, i.e., a glass transition), as well as its relations, via RG, to the directed polymer on a Cayley tree,[14,20,23] and found a nontrivial structure of the strong disorder phase with "quasilocalized" behavior. Interesting relations to the Liouville theory, conjectured in[21] were reexamined and it was found that the freezing transition can be directly demonstrated from renormalization in the Liouville model.[17]

The known results about the exact $E=0$ eigenstate[22] do not, however, tell anything directly about the density of states. In particular the dynamical exponent has not yet been calculated in the strong disorder regime, and one would guess that it should exhibit some kind of change at the transition $\sigma=\sigma_{th}$. A freezing in the dynamical exponent was indeed demonstrated recently[24] in a closely related model, i.e., the classical Arrhenius diffusion in the potential $U(x)$, in both one and two dimension at the same value $\sigma=\sigma_{th}$ than the $E=0$ eigenstate transition. In one dimension the square of the Dirac Hamiltonian is well known to be identical to the

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Fokker-Planck operator and there the two problems are thus equivalent.[25] Thus in one dimension, if one considers a log correlated $U(x)$, both problems have *identical* dynamical exponents and freezing transitions given in Ref. 24. In two dimensions, as discussed below, the two differ by an additional imaginary random drift term, but still they both have line of fixed points and it is reasonable that they would both undergo freezing transitions, as we find here.

In this paper we start by defining the models (Sec. II) and by showing that the density of states (DOS) of the Dirac Hamiltonian can be expressed as an observable in a boson formulation. For convenience we study the DOS $\rho_\epsilon(E)$ at energy $E=0$ adding a small but finite imaginary $i\epsilon$ term for the retarded propagator, thus in effect computing a smoothed DOS, and carefully study the limit $\epsilon\to 0^+$ (Sec. III). At $E=0$ the model becomes very similar, in the boson formulation, to the random gauge XY model in the phase where vortices are relevant. The parameter ϵ plays the role of a bare vortex fugacity and the local DOS $\rho_\epsilon(0,\mathbf{r})$ corresponds to the renormalized vortex fugacity $z_\pm(\mathbf{r})$ (or the local density) that becomes broadly distributed when $\epsilon\to 0$. We show that the order of limits $\epsilon\to 0$ and system size $L\to\infty$ is significant. For $L\to\infty$, such that the typical level spacing $\Delta E<\epsilon$, we use a variational scheme and show that $\rho_\epsilon(0)\sim\epsilon^{2/z-1}$ with z exhibiting a transition at a critical value of disorder. This is equivalent to a phase transition in $\rho(E)\sim E^{2/z-1}$. For $\Delta E>\epsilon$ we find that $\rho_\epsilon(0)$ becomes analogous to the partition function of a directed polymers on a Cayley tree, and also exhibits the freezing transition. It is, however, a strongly fluctuating quantity in that limit and is interpreted as a typical value, rather than a disorder average. Further analogies with freezing of dynamical exponents in Arrhenius dynamics is presented. Finally, in Sec. IV we include a scalar random potential with variance δ, as in the full quantum Hall system. We find that the DOS is noncritical in E, however, its σ dependence is critical, i.e., $\rho(0)\sim\delta^{(2-z)/z'}$. We also develop a variational scheme for studying the transport and localization exponents.

II. SINGLE-LAYER MODEL, DEFINITIONS, AND EXACT MAPPINGS

Our aim is first to study the density of states of the random Dirac Hamiltonian in two space dimensions

$$H_D=\hbar v_F\boldsymbol{\tau}\cdot[-i\boldsymbol{\nabla}-\mathbf{A}(\mathbf{r})]+W(\mathbf{r}), \tag{1}$$

where $\mathbf{r}=(x,y)$ is the two-dimensional (2D) space, $\boldsymbol{\tau}=(\tau_x,\tau_y)$ are Pauli matrices, $W(\mathbf{r})$ is a random scalar potential, and $\mathbf{A}(\mathbf{r})$ is a random vector potential (in units of $e/\hbar$), both Gaussian with short-range correlations (in the following we set $\hbar v_F=1$). $\mathbf{A}$ can be chosen purely transverse $\mathbf{A}_x=\partial_y V$, $\mathbf{A}_y=-\partial_x V$, and its potential has logarithmic correlations $\overline{[V(\mathbf{r})-V(\mathbf{r}')]^2}\sim\sigma\ln|\mathbf{r}-\mathbf{r}'|$, which defines σ. Two exact zero-energy (unnormalized) eigenstates are then $\psi=(e^V,0)$ and $\psi=(0,e^{-V})$.

Since we are also interested in the local density of states in a given sample and that this is a fluctuating quantity it is convenient to define the smoothed local density of states $\rho_\epsilon(E,\mathbf{r})$ as

$$\rho_\epsilon(E,\mathbf{r})=\frac{1}{\pi}\,\mathrm{Im}\left\langle\mathbf{r}\left|\frac{1}{E-H_D-i\epsilon}\right|\mathbf{r}\right\rangle$$
$$=\frac{1}{\pi}\sum_n\frac{\epsilon}{(E-E_n)^2+\epsilon^2}|\psi_n(\mathbf{r})|^2 \tag{2}$$

where E_n are energy eigenvalues of H_D and ψ_n the eigenstates. For $\epsilon\to 0^+$ one recovers the standard local DOS, and for finite ϵ each level is broaden by Lorentzian. The standard DOS is then defined as the spatial average, for a system of linear size L

$$\rho_\epsilon(E)=\frac{1}{L^2}\int d^2\mathbf{r}\rho_\epsilon(E,\mathbf{r}). \tag{3}$$

Although this usually becomes a smooth function of E for $L\to\infty$ in any given sample, at finite size and for small ϵ it is a series of peaks whose locations usually fluctuate strongly from sample to sample. Clearly these fluctuations are smoothed when ϵ becomes of the order or larger than the typical level spacing ΔE. Naively, if the lowest energy states scale as L^{-z} then dimensional argument gives for the space averaged DOS $\rho_\epsilon(0)\sim L^{z-2}$ for $\epsilon<L^{-z}$ or $\rho_\epsilon(0)\sim\epsilon^{2/z-1}$ for $\epsilon>L^{-z}$.

The local DOS can be expressed from the free fermion action with spinors $\overline{\psi}(\mathbf{r})$, $\psi(\mathbf{r})$, projected into a subspace of energy E that defines the Dirac problem in $2+1$ dimensions.

$$\rho_\epsilon(E,\mathbf{r})=\frac{1}{\pi}\,\mathrm{Im}\langle\overline{\psi}(r)\psi(r)\rangle_{\mathcal{S}_D}, \tag{4}$$

$$\mathcal{S}_D=\int d^2r\,\overline{\psi}(\mathbf{r})\{\boldsymbol{\tau}\cdot[-i\boldsymbol{\nabla}-\mathbf{A}(\mathbf{r})]+W(\mathbf{r})-E+i\epsilon\}\psi(\mathbf{r}). \tag{5}$$

An additional Dirac mass term $\Delta_d\tau_z$ in Eq. (5) controls the distance from criticality and is set here to zero.

The problem can be mapped onto a sine-Gordon model. Considering y as an imaginary time variable this action can be written as a (1+1)-dimensional fermion problem. Further bosonization[1] yields the action

$$\mathcal{S}_B=\int d^2r\left\{\frac{1}{8\pi K}[\boldsymbol{\nabla}\theta(\mathbf{r})]^2+\frac{i}{2\pi}[A_y(\mathbf{r})\partial_x-A_x(\mathbf{r})\partial_y]\theta(\mathbf{r})\right.$$
$$\left.-\frac{i}{\pi\alpha}[W(\mathbf{r})-E+i\epsilon]\cos\theta(\mathbf{r})\right\}, \tag{6}$$

where α (which denotes $\hbar v_F\alpha$) is the momentum cutoff and $K=1$; $K\neq 1$ may be generated by RG or correspond to 1D-type interactions. Allowing for $K\neq 1$ is mainly instructive as it allows to interpolate towards the random gauge XY model, and we call this situation the generalized Dirac model. In Eq. (6) $\theta(\mathbf{r})$ is a nonsingular phase field, which can have solitons (but no vortices) generated by disorder. Equation (6) can also

be derived from the network model of the QHE (Refs. 9–11) where x is discretized, $W(x,\tau)$ is then the long-range component of the random potential while $A_y(x,\tau)=(-)^x W(x,\tau)$ is the short wavelength component; both terms couple to slowly varying fields, hence can be considered as independent random variables, though with equal averages.

In the sine-Gordon formulation, the local (smoothed) DOS is given exactly by an average of the operator $\cos(\theta)$ as follows:

$$\rho_\epsilon(E,\mathbf{r})=-\frac{1}{\pi L^2}\frac{\delta}{\delta W(\mathbf{r})}\mathrm{Im}\ln Z=\frac{1}{\pi^2\alpha}\mathrm{Re}\langle\cos\theta(\mathbf{r})\rangle, \tag{7}$$

where $Z=\int D\bar{\psi}D\psi e^{-S_D}$ and $\langle\cos\theta\rangle$ is an average over θ with the action Eq. (6).

We note that all the above mappings are exact. They are even exact for a finite-size sample with some specified boundary conditions for the path integrals. Here we will not need to detail the correspondences in boundary conditions, but it can be done in principle. Note that since the action is complex, the $\langle\cos\theta(\mathbf{r})\rangle$ can be arbitrarily large (when the denominator vanishes), which is the case for $\epsilon\to 0^+$ as E crosses an eigenvalue E_n.

III. RANDOM VECTOR-POTENTIAL MODEL

We now set $W(\mathbf{r})=0$ and study the model with only a random vector potential. To determine the dynamical exponent z we will study the smoothed DOS at zero energy $E=0$. Below we will distinguish two limits and study them separately. First in the large-size limit ($\epsilon>L^{-z}$), if we assume, as is customary, that there is a well-defined density of states $\rho(E)\sim E^{2/z-1}$, when $L\to+\infty$ one has

$$\rho_\epsilon(0)=\frac{1}{\pi}\int dE\rho(E)\frac{\epsilon}{\epsilon^2+E^2}\sim\epsilon^{2/z-1}, \tag{8}$$

for fixed small ϵ and $z>1$ (we will see below that for Dirac fermions $K=1$, the exponent z is indeed larger than unity for all $\sigma>0$). Thus we can obtain z unambiguously from $\rho_\epsilon(0)$. This observable should be self-averaging in that limit since the DOS at zero energy receives contributions from many energy levels in a window of size ϵ around $E=0$, and this is what we find below.

There is another interesting limit, also studied below, when $\epsilon<L^{-z}$ is small (respectively finite size). Then there are fewer energy levels and $\rho_\epsilon(0)$ becomes a strongly fluctuating quantity, as discussed below, which gives information about the statistics of the lowest energy levels E_0 near $E=0$, and thus also about the typical energy-level spacing, found to scale also as $E_0\sim L^{-z}$ with the same exponent z.

A. Large-size limit

A look at Eq. (6) setting $E=0$ [with $W(\mathbf{r})=0$] shows that the model is *identical* to the sine-Gordon (or equivalently Coulomb-gas) formulation[16,18] of the random gauge XY model. Clearly ϵ plays the role of the bare vortex fugacity, but as shown in Ref. 16 the random vector potential generates, upon coarse graining, an additional local random potential resulting in a random fugacity $z_\pm(\mathbf{r})=\epsilon e^{\pm U(\mathbf{r})}$ for ± 1 charges. Some of the physics of the random gauge XY model will thus be relevant here. In particular the local DOS $\rho_\epsilon(0,\mathbf{r})$ is analougous to the coarse grained $z_\pm(\mathbf{r})$ and thus become broadly distributed as $\epsilon\to 0$, as discussed below.

It is convenient to perform a replica average on $\mathbf{A}$ in Eq. (6). This yields a Hamiltonian for the replicated field $\theta_a(\mathbf{r})$ with replica indices $a,b=1,\ldots,m$

$$\mathcal{H}=\int d^2r\left\{(1/8\pi)\sum_{ab}(K^{-1}\delta_{ab}+\sigma)\nabla\theta_a\cdot\nabla\theta_b-\sum_{\mathbf{n}}Y[\mathbf{n}]\exp(i\mathbf{n}\cdot\boldsymbol{\theta})\right\}. \tag{9}$$

where $\langle A_x^2(q)\rangle=\langle A_y^2(q)\rangle=\pi\sigma$, $\mathbf{n}$ is a vector of length m with entries 0, ± 1, and $Y[\mathbf{n}]\sim\Pi_a\epsilon^{n_a^2}$. The term $\epsilon\cos\theta$ in Eq. (6) corresponds to $Y[\mathbf{n}]$ with $\Sigma_a n_a^2=1$ while all other $\mathbf{n}$ are generated by RG. The inclusion of all these terms is essential for treating properly the strong disorder situation,[15,16,18] and obtaining the correct scaling dimension of the $\epsilon\cos\theta$ operator, that is what we need here. Since ϵ is finite, and we are mostly interested in the region $K>1/2$ where the vortices are relevant, they will exist in finite density (separated by a scale ϵ^{-z}). Since we are interested in the end in the behavior as $\epsilon\to 0$ (dilute limit) we can use the RG method developed in Ref. 16 and follow the full distribution of fugacities or equivalently all the $Y[\mathbf{n}]$, up to the length scale at which the vortices separation becomes of order 1, corresponding to $L\sim\sigma_c^{-1/2}$ below (see Ref. 19 for a similar RG study).

For simplicity we use instead a variational method, shown in our previous studies[18] to be good enough to describe the dilute vortex system. In the limits of interest in Secs. III and IV A (small E,ϵ) this variational method is easily seen (by comparison to the above-mentioned RG) to give the *exact result* for scaling dimensions. Very much as in Ref. 16 we expect the more precise RG treatment to correct only amplitudes at weak disorder, powers of logarithmic prefactors at strong disorder and be necessary mostly for detailed descriptions very near the transitions, which we leave for future publication. The results given below for the exponent z and z' should thus be considered as exact.

The variational $\mathcal{H}_0$ has the form

$$\mathcal{H}_0=\int d^2r\left\{(1/8\pi)\sum_{ab}[(K^{-1}\delta_{ab}+\sigma)\nabla\theta_a\cdot\nabla\theta_b+(\sigma_c\delta_{ab}+\sigma_0)\theta_a\theta_b]\right\} \tag{10}$$

with σ_c, σ_0 variational mass parameters. The propagators of the θ field are used to define

$$\sum_{\mathbf{q}}\langle\theta_a(\mathbf{q})\theta_b(-\mathbf{q})\rangle_0=-2u\delta_{ab}-A,$$

$$u=-(K/2)\ln(\Delta_c/4\pi K\sigma_c),$$

$$A=\sigma K^2\ln(\Delta_c/4\pi K\sigma_c)+K\sigma_0/\sigma_c-\sigma K^2 \tag{11}$$

and $\Delta_c \gg \sigma_c$ is an integration cutoff. The interaction term is ($\epsilon/\pi\alpha\to\epsilon$ here)

$$\left\langle \sum_{\mathbf{n}} Y[\mathbf{n}]\exp(i\mathbf{n}\cdot\boldsymbol{\theta})\right\rangle = \left\langle \sum_{\mathbf{n}} \exp\left[(u+\ln\epsilon)\sum_a n_a^2+\omega\sum_a n_a\right]\right\rangle_\omega \tag{12}$$

where the ω average reproduces the required form with $A\Sigma_a n_a^2$,

$$\langle\cdots\rangle_\omega=\int\cdots\exp[-\omega^2/2A]d\omega/(2\pi\sqrt{A}). \tag{13}$$

and has the physical interpretation of an average over random local fugacities.[16,18] The sum in Eq. (12) can be written as $\langle\mathcal{H}_{int}\rangle_0=\langle Z^m\rangle_\omega$ with

$$Z=1+\epsilon e^{u+\omega}+\epsilon e^{u-\omega}. \tag{14}$$

The variational free energy is then minimized, $F_{var}=F_0+\langle H-H_0\rangle_0$, where F_0 is the free energy of Eq. (10) and $\langle\rangle_0$ is an average with weights $\exp(-\mathcal{H}_0)$. This procedure yields[18] an equation for σ_c

$$\sigma_c=\int\frac{\epsilon e^{u+\omega}+\epsilon e^{u-\omega}+4\epsilon^2e^{2u}}{(1+\epsilon e^{u+\omega}+\epsilon e^{u-\omega})^2}e^{-\omega^2/2A}\frac{d\omega}{\sqrt{2\pi A}}. \tag{15}$$

Equation (15) can be solved by steepest descent when the logarithms are large. A similar equation for σ_0 yields that σ_0/σ_c in Eq. (11) is at most finite and, therefore, can be neglected to determine exponents. The result is a phase diagram shown in Fig. 1 with a massive phase $\sigma_c\neq 0$ bounded by the lines $2-K+\sigma K^2=0$ and $\sigma=1/8$. Furthermore, the line $\sigma=2/K^2$ manifests a phase transition corresponding to a change in the relation $\sigma_c\sim\epsilon^{2/z}$ where

$$z=2-K+\sigma K^2,\quad \sigma<2/K^2,$$

$$z=K(\sqrt{8\sigma}-1),\quad \sigma>2/K^2. \tag{16}$$

This transition occurs as rare regions of the sample rather than typical ones start dominating the behavior,[16] as can be also seen from Eq. (15). For $\sigma<2/K^2$ one can discard denominators (as well as the ϵ^2 term) that immediately yields Eq. (16). For $\sigma>2/K^2$ the average over the random fugacity ω is dominated by the tail of the distribution and the right-hand side of Eq. (15) can be approximated by:[26]

$$\mathrm{Prob}(\omega+u+\ln\epsilon>0)\approx\exp\left(\frac{\left(K+2\dfrac{\ln\epsilon}{\ln\sigma_c}\right)^2}{8\sigma K^2}\ln\sigma_c\right) \tag{17}$$

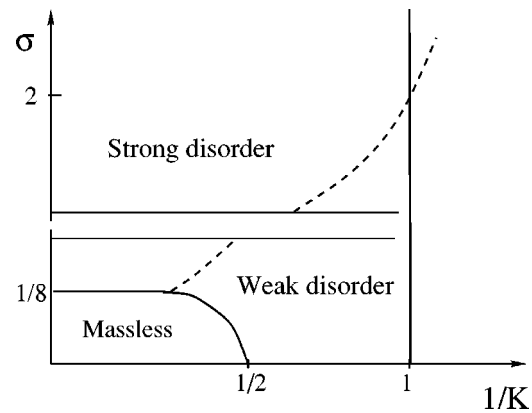

FIG. 1. Schematic phase diagram for the generalized Dirac problem (e.g., single-layer disordered Coulomb gas). σ is the strength of the random gauge disorder. K is an interaction parameter so that $K=1$ corresponds to free fermions. The full line is the phase boundary above which single charges become relevant and below which the model is massless. The dashed-line indicates the freezing transition between weak and strong disorder regimes. Although the K and σ axis here are, strictly, renormalized values, these can be set to bare ones whenever we are interested in the dominant behaviors as $\epsilon\to 0$.

yielding the strong disorder form of Eq. (16). Note that the thermodynamic relation of σ_c and ϵ, and thus Eq. (16) for the Dirac problem, is valid only in the massive phase (its more general meaning is discussed below).

The above results now allow to compute straightforwardly the *disorder averaged* DOS. Indeed we can identify the disorder average of Eq. (2) as $\overline{\rho_\epsilon(0)}=(\partial/\partial\epsilon)F_{var}/\pi L^2$ with the overline denoting average disorder. This yields the replica average

$$\overline{\rho_\epsilon(0)}=\frac{1}{\pi}\frac{\partial}{\partial\epsilon}\left\langle\sum_{\mathbf{n}} Y[\mathbf{n}]\exp(i\mathbf{n}\cdot\boldsymbol{\theta})\right\rangle. \tag{18}$$

Using Eq. (15) for σ_c and a corresponding equation for σ_0 we find $\overline{\rho_\epsilon(0)}\sim(\sigma_c+\sigma_0)/\epsilon\sim\epsilon^{2/z-1}$. Note that Eq. (18) has beyond the $\langle\cos\theta\rangle$ term all the higher-order terms in ϵ as generated by RG. The result differs from just the $\langle\cos\theta\rangle$ average in the strong disorder regime $\sigma>2/K^2$.

In this derivation we have used that L is large compared with the correlation length $1/\sqrt{\sigma_c}\sim\epsilon^{-1/z}$ so that integration cutoffs are determined by σ_c. Hence our result is that

$$\overline{\rho_\epsilon(0)}\sim\epsilon^{2/z-1}\quad \epsilon>L^{-z}, \tag{19}$$

exhibits a phase transition[27] at $\sigma=2/K^2$. From Eq. (8) this also implies that the DOS also exhibits a transition with $\overline{\rho(E)}\sim E^{2/z-1}$. The condition $\epsilon>L^{-z}$ can be interpreted as the typical level spacing $\Delta E=1/\overline{\rho_\epsilon(0)}L^2$ being small, $\Delta E<\epsilon$, so that levels overlap and the DOS is smooth at finite L.

Our conclusion for the free random Dirac Hamiltonian (1), obtained by setting $K=1$, is that there is a phase transition at $\sigma=2$. We find

$$\rho(E)\sim E^{(1-\sigma)/(1+\sigma)}\quad \text{for}\ \sigma<2, \tag{20}$$

$$\rho(E)\sim E^{(3-\sqrt{8\sigma})/(\sqrt{8\sigma}-1)} \quad \text{for} \quad \sigma>2. \tag{21}$$

The result in the weak disorder phase, $\sigma<2$, coincides with the exact result obtained in Ref. 1, as it should. The value obtained for the threshold coincide with the one for the exact zero-energy eigenstate.[20,22] The main novelty here is that we show that at this value of disorder a sharp change of behavior also occurs in the DOS and we obtain the exact result in the strong disorder phase $\sigma>2$. A more detailed treatment reveals that in the strong disorder phase there are logarithmic prefactors to the DOS, as was also the case in Ref. 24. For $\sigma>2$ one has

$$\rho(E)\sim E^{2/z-1}|\ln E|^{-\psi}, \tag{22}$$

where we find for $E=i\epsilon$ that $\psi=\frac{1}{2}\gamma[\sqrt{8\sigma}/(\sqrt{8\sigma}-1)]$, with $\gamma=1/2$ (see Ref. 26). Finally, note that since we are studying only the dominant behavior of the DOS as $\epsilon\to 0$ the σ appearing in the above formulas can be set to be the bare one, since taking perturbative corrections into account yields only subdominant corrections (see below).

As mentioned above, using the RG of Ref. 16 yields the same result (16) for the dynamical exponent z. This is a result about the true scaling dimension z of the $\epsilon\cos\theta$ operator (noted Δ_{typ} in the conclusion of Ref. 16) being different, in the strong disorder regime, from the naive one (noted Δ there), as occurs in the XY model (see discussion there). This transition in the scaling dimension is a property of the $\epsilon=E=0$ theory and holds whether or not the operator itself is relevant (sign of the scaling dimension) in the (massless) XY phase of the XY model where it is irrelevant, it still corresponds to a true phase transition, but only for the single vortex problem[17] and not for the full XY model. Finally note that the RG treatment is expected to change the value of the exponent ψ of the logarithmic corrections (i.e., γ is expected to change to $\gamma=3/2$).[17]

B. Finite-size regime

Let us now characterize some aspects of the fluctuations of the DOS in a finite-size system. Within the variational method described above one sees that for $\epsilon>L^{-z}$ the system is too small to generate the mass σ_c hence $\rho_\epsilon(0)\sim\langle\cos\theta\rangle_0$ with $\sigma_c=\sigma_0=0$ in Eq. (10), i.e.,

$$\overline{\rho_\epsilon(0)}\sim L^{-K+\sigma K^2}, \quad \epsilon<L^{-z}. \tag{23}$$

Since the system is effectively massless we expect significant fluctuations. In the following we consider a different approach for the $\epsilon<L^{-z}$ case that will clarify the nature of disorder average.

We proceed to evaluate the DOS by a direct expansion in ϵ. At $\epsilon=0$ a direct evaluation of the Gaussian average over θ in a given sample [assuming periodic boundary conditions for the resulting potential $V(\mathbf{r})$] yields[28]

$$\langle\cos\theta(\mathbf{r})\rangle=e^{-K\ln L}[e^{-U(\mathbf{r})}+e^{U(\mathbf{r})}]/2 \tag{24}$$

where, in Fourier space, $U(q)=2KV(q)=(2K/q^2)(iq_xA_y-iq_yA_x)$ with correlation of the form

$$\overline{[U(\mathbf{r})-U(0)]^2}=4\sigma K^2\ln r. \tag{25}$$

The density of states thus takes the form

$$\rho_{\epsilon=0}(0)=\frac{1}{L^2}\sum_{\mathbf{r}} e^{-K\ln L}[e^{-U(\mathbf{r})}+e^{U(\mathbf{r})}]/2=\frac{1}{L^2}Z_L \tag{26}$$

of the partition function Z_L of a single $\pm$ vortex in a logarithmically correlated random potential, known to be related to the one of a directed polymer on a Cayley tree.[14,17] A simple average of the partition function Z_L, i.e., $\overline{\rho_\epsilon}(0)$ indeed leads to Eq. (23), however, as is well known in the directed polymer problem only the logarithm of the partition function $\ln Z_L$ is self-averaging. This immediately yields[17,29,30] our result for the typical DOS at finite size

$$\rho_{typ}(0)\sim L^{-2+z}, \tag{27}$$

with z given by Eq. (16).

To identify the role of ϵ we consider the first-order terms $\epsilon\int d^2r'\langle\exp[\pm\theta(\mathbf{r})\pm\theta(\mathbf{r}')\rangle$. The typical value of each of these terms scales as the typical value of Z_L^2/L^2 (for opposite charges it is true in the massive phase we are interested in). This allows to identify a crossover function $f(x)$, where

$$\rho_{typ}(0)\sim L^{-2+z}+\epsilon L^{-2+2z}+\cdots=L^{-2+z}f(\epsilon L^z), \tag{28}$$

with $f(x)=1+x$ at $x\to 0$. For $x\gg 1$ we can recover Eq. (19) if the crossover function satisfies $f(x)\sim(1/x)^{1-2/z}$, i.e., the typical value $\rho_{typ}(0)$ crosses over to the average $\overline{\rho_\epsilon}(0)$ at $\epsilon>L^{-z}$, with both limits exhibiting a phase transition.

This statistics can be described in a simple phenomenological picture. A finite ϵ provides a length scale (the vortex separation) and in effect cuts the system in independent pieces of sizes $L_\epsilon=\epsilon^{-z}$. One has thus roughly

$$\rho_\epsilon(0)=\frac{1}{L^2}\sum_{i=1}^{L/L_\epsilon} Z_{L_\epsilon}^{(i)}, \tag{29}$$

where the random variables $Z_{L_\epsilon}^{(i)}$ are independent with a lognormal distribution. For large L/L_ϵ, however, the above sum acquires a normal distribution. A similar picture was used to describe the related random diffusion problem, where the local first-passage times are analogous to the local DOS in the present problem, and an external force produces a finite length scale. Analysis of the various regimes is described there and are expected to be quite similar here.

C. Relation to random diffusion models

It is instructive to compare our results to the one obtained for random diffusion problems. As mentioned in the introduction, general random Dirac problems can be mapped onto random diffusion operators, which in general may involve two species, with absorption, creation, and transformation. It is particularly simple in the case $W=0$ (random vector potential alone) where it maps onto a random Fokker-Planck diffusion operator of the type

$$H_{FP}P=\nabla^2P-\boldsymbol{\nabla}\cdot(\mathbf{F}_T+\mathbf{F}_L)P=-E'P, \tag{30}$$

which describes the Langevin diffusion of a particle, $\partial_t P = H_{FP} P$, where $P(\mathbf{r})$ is the probability that the particle is at point $\mathbf{r}$ at time t, in the presence of both a potential random force $\mathbf{F}_L = -\boldsymbol{\nabla} U$ and a transverse one (a random drift), with div $\mathbf{F}_T = 0$. Equivalently, setting $P = e^{-U/2}\psi$,

$$H'_{FP}\psi = [\nabla^2 - \mathbf{F}_T \cdot \boldsymbol{\nabla} - (\boldsymbol{\nabla} V)^2 + \nabla^2 V]\psi = -E'\psi, \quad (31)$$

with $V = U/2$ ($K=1$ here). The operators H_{FP} and H'_{FP} have the same spectrum. In two dimension, taking the square of Eq. (1) with $W=0$ yields

$$-H_D^2 = \nabla^2 - (A_x^2 + A_y^2) + \sigma_z(\partial_y A_x - \partial_x A_y) - 2\lambda i \mathbf{A}\cdot\boldsymbol{\nabla} - i\boldsymbol{\nabla}\cdot\mathbf{A} \quad (32)$$

with $\lambda = 1$ identical to H'_{FP} with $A_x = \partial_y V$ and $A_y = -\partial_x V$ (and $V \to -V$ for the other component of the spinor) and $F_T = 2iA$. This is thus Arrhenius diffusion in the random potential U with an additional *imaginary* random drift.[25] The diffusion dynamical exponent z_d associated with H'_{FP} should thus be simply $z_d = 2z$. Note that all the operators obtained by varying λ have identical ground-state wave function, $\psi \sim e^{-V}$ since the additional drift term does vanish in the ground state (in the diffusion context it means that the drift is along equipotentials of U). It is thus reasonable to expect that each of these models are described by a line of fixed points and that they all do exhibit a freezing transition for any value of λ *at the same value* of $\sigma = \sigma_{th} = 2$.

In the absence of this additional drift (i.e., setting $\lambda = 0$), the problem reduces to the one studied in[24] where indeed it was found that there is also a freezing transition in the dynamical exponent z_d in $d=1$ and $d=2$ with (assuming conventional dynamical scaling)

$$z_d(\lambda=0) = 2 + 2(\sigma/\sigma_{th}), \quad \sigma < \sigma_{th}, \quad (33)$$

$$z_d(\lambda=0) = 4\sqrt{\sigma/\sigma_{th}}, \quad \sigma > \sigma_{th}. \quad (34)$$

Although it does indeed exhibit a freezing transition at the same threshold $\sigma_{th} = d$, one sees that $z_d(\lambda=0) \leq z_d(\lambda=1) = 2z$, i.e., the imaginary drift slow down the diffusion, presumably through interference effects. It would be of interest to determine $z_d(\lambda)$ as well as to study freezing transitions in a generalized class of these diffusion models in two dimensions.

It is possible to consider various one-dimensional restriction of the Dirac model, e.g., the so-called supersymmetric quantum mechanics that also exhibits band-center delocalization.[31,32] With a log correlated $U(x)$ this model was studied analytically in Ref. 24, thus we know in that case the exact $z = z_d/2$ dynamical exponent of the random Dirac operator.

IV. FULL QUANTUM HALL PROBLEM

A. One layer problem: scaling of the DOS

Finally, we consider the Dirac model where the scalar random potential in Eq. (5) is retained, which describes the full quantum Hall system. We will first determine the DOS at zero energy, and later around zero energy.

In presence of a random scalar potential one has $Y[\mathbf{n}] \sim (iW - \epsilon)\Sigma n_a^2$ in Eqs. (9) and (18). We can safely set $\epsilon = 0$ in the definition $\rho_\epsilon(0) \sim \partial F_{var}/\partial\epsilon$ since $W(\mathbf{r})$ provides a mass parameter. The variational method is similar to the previous case, except that ϵ is replaced by $iW(\mathbf{r})$ in Eq. (15). Since the integral is dominated by large u and ω (for $\omega > 0$), it has the form

$$\sigma_c = \left\langle \frac{iWe^{u+\omega}}{(1+iWe^{u+\omega})^2} \right\rangle_{\omega,W} \quad (35)$$

$$= \left\langle \frac{2W^2 e^{2u+2\omega}}{(1+W^2 e^{2u+2\omega})^2} \right\rangle_{\omega,W}, \quad (36)$$

using the $\pm$ symmetry of the W average.[26] The latter form is equivalent to the previous integral Eq. (16) with ϵ replaced by the disorder average $\langle W^2 \rangle = \delta$ (for the starting QH system $\delta \sim \sigma$) and K is replaced by $2K$. Hence $\sigma_c \approx \delta^{2/z'}$, where now

$$z' = 2 - 2K + 4\sigma K^2, \quad \sigma < 1/2K^2,$$

$$z' = 2K(\sqrt{8\sigma} - 1), \quad \sigma > 1/2K^2. \quad (37)$$

The DOS at zero energy can be written as

$$\overline{\rho(E=0)} \sim \langle \cos\theta \rangle \sim \left\langle \frac{e^{u+\omega}}{1+W^2 e^{2u+2\omega}} \right\rangle_{\omega,W} \sim \delta^\alpha, \quad (38)$$

where

$$\alpha = \frac{2}{z'} - \frac{z}{z'}. \quad (39)$$

Since z' has a transition at $\sigma = 1/2K^2$ the DOS has *two transitions*, at $\sigma = 1/2$ and at $\sigma = 2$ (for $K=1$). The exponent α in Eq. (39) is the one expected from a scaling form $\rho_{\epsilon,\delta}(0) = \delta^\alpha g(\epsilon/\delta^{z/z'})$, which connects with the $\delta = 0$ case solved in Sec. III [which requires $g(x) \to 1$ at $x \to 0$ and $g(x) \sim x^{\alpha z'/z}$ at $x \to \infty$]. As we will see below there are, however, three phases, each with a different scaling function $g(x)$. Note that z'/z increases from 0 at small σ to $z'/z = 2$ at $\sigma = 2$ and remains equal to this value for stronger disorder.

We have thus shown that the DOS is finite at the quantum Hall transition, which is a well-established result.[33] In addition, however, we have determined the scaling of the zero-energy density of states with the scalar potential strength in the random Dirac problem. Note that in the previous case with $\epsilon \to 0$ renormalization of K and σ was of higher order in ϵ and could be neglected. Here, however, a finite $\langle W^2(\mathbf{r}) \rangle$ renormalizes both K and σ, with σ flowing to stronger values. This does not spoil our result for the exponent z' in the limit $\delta \to 0$, however, as it would also yield only subleading corrections in δ.

The physics of the problem becomes more apparent when one notes that the random scalar potential produces [see e.g., Eq. (6)] a random (imaginary) fugacity with a random sign. One can then again either extend the RG of Ref. 16 to this situation or consider the extreme dilute limit (single vortex)

as in Eq. (24). Both considerations lead immediately to a mapping onto the directed polymer on the Cayley tree with random Boltzmann weights *of random sign* (in the bare model the random sign acts only on the leafs of the tree but since both signs have equal probability one easily sees that this is equivalent to a random sign on each branch of the tree). This model was also solved in Ref. 34 and is known to indeed exhibit a transition at half the value of the same sign problem, due to interference effects that in effect bind two replicas. The value of the z' exponent obtained by this method is identical to the one given above [Eq. (37)]. It is remarkable, and encouraging, that the variational method also captures this physics.

We can now extend these considerations and obtain, in implicit form, the crossover function that describes the DOS $\rho(E)$ in the small δ limit. We will obtain explicitly $\overline{\rho_\epsilon(0)}$, from which $\overline{\rho(E)}$ can be extracted inverting Eq. (8).

The equation that determines σ_c and the DOS is now

$$\sigma_c=\left\langle \int d\omega \frac{\epsilon e^{u+\omega}+(W^2+\epsilon^2)e^{2u+2\omega}}{1+2\epsilon e^{u+\omega}+(W^2+\epsilon^2)e^{2u+2\omega}}\right\rangle_{\omega,W}, \tag{40}$$

$$\overline{\rho_\epsilon(0)}=\left\langle \int d\omega \frac{e^{u+\omega}+\epsilon e^{2u+2\omega}}{1+2\epsilon e^{u+\omega}+(W^2+\epsilon^2)e^{2u+2\omega}}\right\rangle_{\omega,W}, \tag{41}$$

the first line is really $\sigma_c+\sigma_0$ but we can use that σ_0 is subdominant and only dominant exponents for the $\omega>0$ are retained.[26] We immediately see that there are several regimes, according to whether one can neglect all denominators (weak disorder), or whether the averages will be dominated by the rare events where either the terms proportional to W^2 or to ϵ or both, are of order 1. There are in fact three phases, in each of them scaling holds with different scaling functions f, g, $\mathcal{R}$,

$$\sigma_c=\delta^{2/z'}f(\epsilon/\delta^{z/z'}), \tag{42}$$

$$\overline{\rho_\epsilon(0)}=\delta^{(2-z)/z'}g(\epsilon/\delta^{z/z'}), \tag{43}$$

from which one can extract the DOS scaling function

$$\rho(E)=E^{2/z-1}\mathcal{R}(E/\delta^{z/z'}) \tag{44}$$

determined implicitly by the relation

$$g(x)=\frac{1}{\pi}\int dy y^{2/z-1}\mathcal{R}(y)\frac{x}{x^2+y^2}. \tag{45}$$

Weak disorder phase $\sigma<1/2K^2$. Neglecting all denominators the equation for σ_c and $\overline{\rho_\epsilon(0)}$ become

$$\overline{\rho_\epsilon(0)}=\sigma_c^{1-z/2}, \tag{46}$$

$$1=\epsilon\sigma_c^{-z/2}+\delta\sigma_c^{-z'/2}, \tag{47}$$

which yield the scaling functions in implicit form

$$x=f(x)^{z/2}[1-f(x)^{-z'/2}], \tag{48}$$

$$g(x)=f(x)^{1-z/2}, \tag{49}$$

possibly the exact ones, up to prefactors.

Strong disorder phase I, $1/2K^2<\sigma<2K^2$. There one can still neglect denominators in averages involving the terms proportional to ϵ but not in the one involving the terms proportional to W^2. One finds

$$\overline{\rho_\epsilon(0)}\sim\sigma_c^{1-z/2}, \tag{50}$$

$$\sigma_c=\epsilon\sigma_c^{1-z/2}+\sigma_c^{(K+[\ln\delta/\ln\sigma_c])^2/(8\sigma K^2)}. \tag{51}$$

Simple expansion shows that scaling still holds but the scaling functions are now implicitly given by

$$x=f(x)^{z/2}[1-f(x)^{-\tilde{z}'/2}], \tag{52}$$

$$\tilde{z}'=4\left(1-\frac{1}{\sqrt{8\sigma}}\right),$$

$$g(x)=f(x)^{1-z/2}.$$

This scaling function is accurate only up to logarithmic prefactors. A more accurate form is $\sigma_c\sim\delta^{2/z'}|\ln\delta|^{-2\gamma/\tilde{z}'}f(\epsilon\delta^{-z/z'}|\ln\delta|^{2\gamma/\tilde{z}'})$ with $\gamma=1/2$ but γ is likely to be corrected upon a more careful RG treatment.

Strong disorder phase II, $2/K^2<\sigma$. At even stronger disorder, since $z'=2z$, we expect the scaling region to be $\epsilon^2\sim\delta$. To show that this is the case and to get some approximation for $f(x)$ we notice that the equation for σ_c can be approximated in the scaling region by

$$\sigma_c\approx\mathrm{Prob}(\epsilon e^{u+\omega}+(\epsilon^2+\delta)e^{2(u+\omega)}>1). \tag{53}$$

Solving the quadratic equation in $e^{u+\omega}$ yields that σ_c is indeed of the form (42), with some form for $f(x)$, which here is approximate. We have not attempted to solve more precisely for $f(x)$ or $g(x)$ in this phase.

Finally note that this crossover can also be studied at finite size, and is there complicated as it will probably be described as in Ref. 34 by a nontrivial phase diagram.

B. Transport in the quantum Hall system: the two layer problem

We address now the more difficult problem of describing transport and localization in the QHE system. Transport is derived by a disorder average of advanced and retarded propagators, hence two partition sums corresponding to the action of Eq. (6) with $\pm\epsilon$. The problem is then of two layers with fields $\theta_\uparrow$ and $\theta_\downarrow$ and common disorder $\mathbf{A}(\mathbf{r})$ and $W(\mathbf{r})$. The role of $\pm\epsilon$ is to determine the proper ground state near which the variational method applies, i.e., for $-\epsilon$ we shift $\theta_\downarrow\to\theta_\downarrow+\pi$ so that the nonlinear terms become

$$\exp\left\{-\sum_a\ [(-iW+\epsilon)\cos\theta_{\uparrow,a}+(iW+\epsilon)\cos\theta_{\downarrow,a}]\right\}, \tag{54}$$

where $a=1,\ldots,m$ are replica indices for each layer and we have redefined here $W/\pi\alpha,\epsilon/\pi\alpha\to W,\epsilon$, respectively. Expansion in powers of $iW\pm\epsilon$ and identifying the dominant terms of the form $\exp(i\mathbf{n}\cdot\boldsymbol{\theta})$ yields

$$\mathcal{H}_{int}=\sum_{\mathbf{n}}\ (-iW)^{\Sigma_a n_{\uparrow a}^2}(iW)^{\Sigma_a n_{\downarrow a}^2}\exp(i\mathbf{n}\cdot\boldsymbol{\theta}), \tag{55}$$

with ϵ now set to zero, $\mathbf{n}$ is now a vector of $2m$ entries $(n_{\uparrow 1},\ldots n_{\uparrow m},n_{\downarrow 1},\ldots,n_{\downarrow m})$ and similarly for $\boldsymbol{\theta}$.

The Gaussian part of the Hamiltonian can be written as

$$\mathcal{H}'=\sum_{\mathbf{q},a}\frac{q^2}{8\pi K}[|\theta_{+,a}(\mathbf{q})|^2+|\theta_{-,a}(\mathbf{q})|^2]+\sum_{\mathbf{q},a,b}\frac{\sigma q^2}{4\pi}\theta_{+,a}(\mathbf{q})\theta_{+,b}(-\mathbf{q}), \tag{56}$$

where $\theta_{\pm a}=(\theta_{\uparrow a}\pm\theta_{\downarrow a})/\sqrt{2}$. Since only the θ_+ mode is affected by the common disorder $\mathbf{A}(\mathbf{r})$ we expect $\theta_\pm$ to have distinct self-masses. The variational Hamiltonian is then

$$\mathcal{H}_0=\mathcal{H}'+\frac{1}{2}\sum_{\mathbf{q},\pm,a,b}[\sigma_c^{\pm}|\theta_{\pm a}(\mathbf{q})|^2\delta_{ab}+\sigma_0^{\pm}\theta_{\pm a}(\mathbf{q})\times\theta_{\pm b}(-\mathbf{q})]. \tag{57}$$

The propagators of the $\pm$ modes are used to define

$$\sum_{\mathbf{q}}\langle\theta_{\pm a}(\mathbf{q})\theta_{\pm b}(-\mathbf{q})\rangle_0=-4u_\pm\delta_{ab}-2A_\pm,$$

$$u_\pm=-(K/4)\ln(\Delta_c/4\pi K\sigma_c^{\pm}),$$

$$A_+=\sigma K^2\ln(\Delta_c/4\pi K\sigma_c^{+})+K\sigma_0^{+}/2\sigma_c^{+}-\sigma K^2,$$

$$A_-=K\sigma_0^{-}/2\sigma_c^{-}. \tag{58}$$

We write the interaction Eq. (55) in the form

$$\langle\mathcal{H}_{int}\rangle_0=\left\langle(-iW)^{\Sigma_a n_{\uparrow a}^2}(iW)^{\Sigma_a n_{\downarrow a}^2}\times\exp\left[u_+\sum_a(n_{\uparrow a}+n_{\downarrow a})^2+\omega_+\sum_a(n_{\uparrow a}+n_{\downarrow a})+u_-\sum_a(n_{\uparrow a}-n_{\downarrow a})^2+\omega_-\sum_a(n_{\uparrow a}-n_{\downarrow a})\right]\right\rangle_\omega, \tag{59}$$

where the ω average reproduces the required form with $A_\pm(\Sigma_a n_{\uparrow a}\pm n_{\downarrow a})^2$,

$$\langle\cdots\rangle_\omega=\int\int\exp\left[-\frac{\omega_+^2}{2A_+}-\frac{\omega_-^2}{2A_-}\right]\frac{d\omega_+d\omega_-}{2\pi\sqrt{A_+A_-}}. \tag{60}$$

The sum in Eq. (59) can be written as $\langle\mathcal{H}_{int}\rangle_0=\langle Z^m\rangle_\omega$ with

$$\begin{aligned}Z=&1-iW\exp[u_++u_-+\omega_++\omega_-]\\&+iW\exp[u_++u_-+\omega_+-\omega_-]\\&-iW\exp[u_++u_--\omega_+-\omega_-]\\&+iW\exp[u_++u_--\omega_++\omega_-]+W^2e^{4u_++2\omega_+}\\&+W^2e^{4u_-+2\omega_-}+W^2e^{4u_--2\omega_-}+W^2e^{4u_+-2\omega_+}.\end{aligned} \tag{61}$$

The masses are to be found by minimizing the variational free energy $F_{var}=F_0+\langle H-H_0\rangle_0$, as in Sec. III A. We expect to find a massless solution, e.g., $\sigma_c^-=0$. This is indeed a possible solution with $u_-\to-\infty$ and

$$Z=1+W^2e^{4u_++2\omega_+}+W^2e^{4u_+-2\omega_+}. \tag{62}$$

W^2 can be replaced by its average and then this has the same structure as the single-layer problem of Sec. III A with the replacement $K\to 2K$, i.e. the phase diagram is the same as Fig. 1 with the $1/K$ axis replaced by $1/2K$. The starting line $K=1$ is now tangent to the phase boundary at $\sigma=0$. However, for $\sigma>0$, θ_+ is massive.

To find the QH localization exponent we introduce the mass term that corresponds to

$$\Delta_d(\sin\theta_\uparrow-\sin\theta_\downarrow)=2\Delta_d\cos(\theta_+/\sqrt{2})\sin(\theta_-/\sqrt{2}). \tag{63}$$

Note the opposite signs due to the shift of $\theta_\downarrow$ as required by the sign of ϵ. Since θ_+ is massive, Δ_d can be replaced by $\tilde{\Delta}_d=2\Delta_d\langle\cos(\theta_+/\sqrt{2})\rangle$ and defining $\theta=\theta_-/\sqrt{2}+\pi/2$ leads to an effective Hamiltonian with

$$H_{eff}=\int d^2r\left[\frac{1}{4\pi K}|\boldsymbol{\nabla}\cdot\theta(\mathbf{r})|^2-\tilde{\Delta}_d\cos\theta(\mathbf{r})\right] \tag{64}$$

This system has a correlation length ξ, related to a mass $\xi^{-2}\sim\Delta_d^{2\nu}$, which from first-order RG is $\nu=2/(4-K)$. For the original QH problem with $K=1$ the localization exponent is then $\nu=2/3$ (the numerically known value is ≈ 2.3). We expect, however, that K is RG driven to a different value. This is beyond the variational scheme that gives reliable exponents only for small (W,ϵ) couplings (but arbitrary σ,K), as in the $\epsilon\to 0$ case. E.g., to second order in replicated sine-Gordon RG, the most relevant operator $W\cos(\theta_a+\theta_b)$ yields

$$\frac{dW}{dl}=(2-2K+4\sigma K^2)W,$$

$$\frac{dK^{-1}}{dl}=-2(K-2\sigma K^2)W^2,$$

$$\frac{d\sigma}{dl}=(K-2\sigma K^2)W^2, \tag{65}$$

which shows that indeed K increases at weak disorder.

We have also looked for other solutions of the variational scheme of the form $\sigma_c^-\sim(\sigma_c^+)^\alpha$ and found that only $\alpha=1$ exists. This corresponds to decoupled layers with the phase

diagram of Fig. 1 (K replaced by $2K$). Hence at $K=1$ this is a massive phase and does not correspond to the QH problem.

Our method shows that a subset of the degrees of freedom can form a massless phase and we hope that it will stimulate further progress.

V. CONCLUSION

In conclusion, we have derived an exact formulation of two-dimensional random Dirac fermions in terms of a random sine-Gordon model, or equivalently a disordered Coulomb gas. The DOS of the Dirac fermion system identifies with the expectation value of a cosine operator in the sine-Gordon model (equivalently the charge fugacity in the Coulomb gas) and the dynamical exponent z as its scaling dimension. We found that at zero energy and with random vector potential only the Dirac system maps onto the random gauge XY model with infinitesimal vortex fugacity (dilute limit of the CG). Using methods and results from previous studies of the XY model, we have computed the exact dynamical exponent z for the random vector-potential model at any disorder, and thus obtained the critical behavior of the DOS around zero energy. We found that it exhibits a transition at the same threshold value than the previously known transition in an exact ground-state wave function. It shows that all eigenstates near the band center are affected by this transition. The physics of this transition is closely related to the freezing transition in the XY model in the limit where the vortex core energy is taken very large. As we show here the density of states in a finite-size sample becomes broadly distributed with typical values scaling differently than average ones with the system size. This corresponds to the eigenstates being peaked around some few centers in the sample.

It is likely that similar freezing phenomena are of importance in a broader class of two-dimensional disordered models. They were recently found to occur in random diffusion models, for instance in the problem of random Arrhenius diffusion of a particle in a log arithmically correlated potential. We have analyzed the similarities and differences of the transitions that occur in the dynamical exponents z of both models. As was shown in Ref. 24 strong disorder renormalization group captures the dynamical behavior in the glass phase, which suggests that it could be used to study the present problem as well.

It is of utmost importance to understand what are the consequences of this transition when a random potential is added, corresponding to the quantum Hall system. As a first step in that direction we have determined the scaling of the DOS (which is then finite at zero energy) when the additional random potential is small. There we found two transitions, one at the same value than the random vector-potential model, the other at a much smaller value of disorder. It would be nice to know whether these transition lines extend away from the random vector-potential fixed line. Numerical checks of our predictions as well as a numerical calculation of a glass order parameter (e.g., $\Sigma_x|\psi(x)|^4$) in the full model would help understand these issues.

We believe that such freezing phenomena, originally studied in the context of disordered Coulomb gas and XY models, will also affect a broader class of disordered fermion models in two dimensions. This can be studied systematically by extending the bosonization approach introduced in the present paper, and for instance, searching for all perturbations around the random gauge fixed plane and computing their (nontrivial) scaling dimension (here it was done only for the vortex fugacity operator). In particular it is of interest to know whether the nonlinear σ models studied in Ref. 8 also exhibit freezing phenomena. They are indeed generalizations of the Liouville model that captures the single vortex problem and does exhibit a freezing transition. Results are already known in related cases. For instance it was shown in Ref. 19 that if one adds pinning disorder to the random gauge XY model, the vortex density (DOS) acquires a $\sim\exp(-|\ln\epsilon|^{2/3})$ dependence in the vortex core energy $\ln(1/\epsilon)$ with a nontrivial exponent.

Finally, we have formulated the problem of the transport in the quantum Hall system as two coupled random sine-Gordon models. We have applied the variational method that should capture some of the nonperturbative effects. We have analyzed the system using in the plane of two (dimensionless) parameters and were able to find a massless phase. Though qualitatively encouraging, to obtain the quantitative characteristics of this phase requires to incorporate in a more precise way additional renormalizations of these two parameters. It is tantalizing that possible values of these renormalized parameters (consistent with numerics) seem to lie within the region near the glass phase boundary.

ACKNOWLEDGMENTS

We thank Bernard Derrida and Leon Balents for useful discussions. B. H. acknowledges support by The Israel Science Foundation founded by the Israel Academy of Science and Humanities.

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