Thermal metal in network models of a disordered two-dimensional superconductor

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We study the symmetry class for localization which arises from models of noninteracting quasiparticles in disordered superconductors that have neither time-reversal nor spin-rotation invariance. Two-dimensional systems in this category, which is known as class D, can display phases with three different types of quasiparticle dynamics: metallic, localized, or with a quantized (thermal) Hall conductance. Correspondingly, they can show a variety of delocalization transitions. We illustrate this behavior by investigating numerically the phase diagrams of network models with the appropriate symmetry and show the appearance of the metallic phase.

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The properties of quasiparticles in disordered superconductors have been a subject of much recent interest. Within a mean-field treatment of pairing, the quasiparticles are noninteracting fermions, governed by a quadratic Hamiltonian which may contain effects of disorder in both the normal part and the superconducting gap function. Such Hamiltonians are representatives of a set of symmetry classes different from the three classes which are familiar both in normal disordered conductors and in the Wigner-Dyson random matrix ensembles. A list of additional random matrix ensembles, determined by these new symmetry classes, has been established relatively recently.[1] These additional random matrix ensembles describe zero-dimensional problems and are appropriate to model a small grain of a superconductor in the ergodic limit. In the corresponding higher-dimensional systems from the same symmetry classes, there can be transitions between metallic, localized, or quantized Hall phases for the quasiparticles.[2–4] The associated changes in quasiparticle dynamics must be probed by energy transport or (in singlet superconductors) spin transport, rather than charge transport, since quasiparticle charge density is not conserved.[2] There are various possibilities for this behavior, depending on the particular symmetry class considered. These have been studied theoretically using nonlinear σ model methods,[2] numerically,[3] and in quasi-one-dimensional models.[5] An important question not addressed in such work so far, and which will not be considered here, is whether the self-consistent solution to the gap equation in the presence of disorder affects the universal statistical properties of the ensembles.

In this paper we present extensive numerical results on a symmetry class with a particularly rich phase diagram in two dimensions, class D. The symmetry may be realized in superconductors with broken time-reversal invariance and either broken spin-rotation invariance (as in *d*-wave superconductors with spin-orbit scattering) or spinless or spin-polarized fermions (as in certain *p*-wave states). The nonlinear σ model for class D (Ref. 1) has been shown, in the two-dimensional case, to flow under the renormalization group to weaker values of the coupling constant.[6–9] The coupling constant is proportional to the inverse of the thermal conductivity of the superconductor, and this flow implies that there is a phase in which there is a nonzero (indeed, diverging[7]) density of extended fermion eigenstates at zero excitation energy. A superconductor described by this model would be in a thermal metal phase. We will refer to such a phase simply as a metallic phase. In addition, a phase with localized quasiparticles is a natural possibility and—since time-reversal symmetry is broken—so is one with quantized thermal Hall (or Leduc-Righi) conductance. Our aim in the following is to investigate the appearance of these phases in simple models. Remarkably, we find that the symmetry class alone is not sufficient to determine which phases occur and that different particular forms of disorder result in quite distinct physical behavior.

As our starting point, we take versions of the network model[10] for a single-component fermion, which we specify in detail after first summarizing our findings. Disorder appears in the network model in the form of random scattering phases, and the symmetries of class D restrict scattering phases to the values 0 and π. Within this framework, we discuss three alternative choices. The first of these was introduced in work by Cho and Fisher (CF) (Ref. 11) with the intention of modeling the two-dimensional random bond Ising model (RBIM), which possesses a fermion representation with the symmetries of class D. In fact, as noted subsequently,[12,13] a precise mapping of the Ising model leads to a second version of the network model, which we label RBIM. In both these models, scattering phases with the value π appear in correlated pairs. A third model (also discussed in Ref. 9), which we denote by O(1), arises naturally if one instead takes scattering phases to be independent random variables.

Each model has two parameters: a disorder concentration p $(0 \leqslant p \leqslant 1)$ and a tunneling amplitude α $(0 \leqslant \alpha \leqslant \pi/2)$, which controls the value of the thermal Hall conductance at short distances.[14] Our phase diagram for the CF model in the (p, α) plane is shown in Fig. 1. It contains a region of metallic phase and two distinct localized phases, which can be identified with the ordered and disordered phases of the

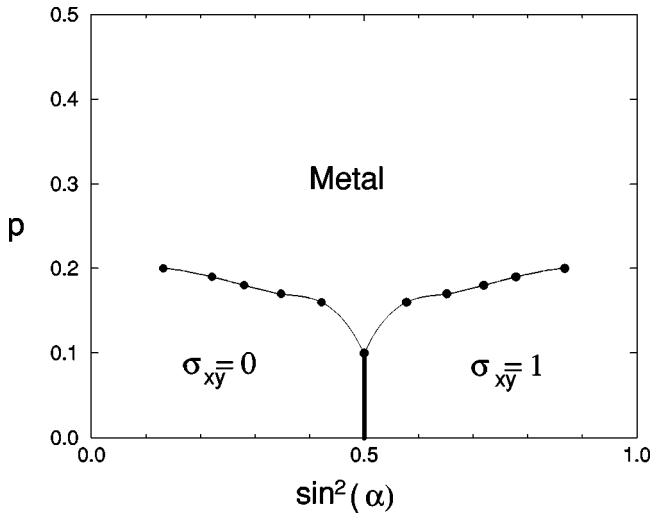

FIG. 1. The phase diagram of the CF model obtained from our numerical calculations.

RBIM or as regions with different quantized thermal Hall conductance. As a consequence, three potentially different critical points occur: an insulator-to-insulator quantum-Hall-type transition, an insulator-to-metal transition, and a multicritical point at which all three phases meet. This phase diagram has the form proposed generically for class D in Ref. 7. In contrast, neither the RBIM nor the O(1) model supports all three phases: arguments that the metallic phase cannot appear in RBIM's with real Ising couplings are given in Ref. 15, while in the O(1) model we find no localized phase, in striking distinction to all network models studied previously. We show below how these differences can be understood by solving the models in one dimension and by considering them in two dimensions at weak tunneling.

All these models represent coherent propagation of quantum-mechanical flux on a square lattice of directed links which meet at nodes, as illustrated in Fig. 2. Such a system can be specified using a discrete-time evolution operator U.[16,14] For a network of $\mathcal{N}$ links, U is an $\mathcal{N}\times\mathcal{N}$ unitary matrix. The action of U on a vector of flux amplitudes, $z_l(t)$, defined on the start of each link l at time t, generates amplitudes $z_l(t+1)$. A matrix element U_{jk} is nonzero only if the links j and k are, respectively, outgoing and ingoing at the same node. Its value in this case is the product of a phase factor $\exp(i\phi_k)$ associated with the link k and a tunneling amplitude associated with the node, which takes the values $\pm\cos(\alpha)$ or $\pm\sin(\alpha)$ as indicated in Fig. 2. Plaquettes of the lattice can be divided into two sets, according to the direction of circulation around them. For general values of α, all plaquettes are coupled, but for $\alpha=0$ the system separates into uncoupled plaquettes with clockwise circulation, while for $\alpha=\pi/2$ it consists of uncoupled anticlockwise plaquettes. Disorder is introduced via the link phases ϕ_l. To make clear the constraints imposed in class D, recall that a Bogoliubov–de Gennes Hamiltonian with this symmetry may be written in terms of a purely *imaginary* Hermitian matrix.[1] The corresponding time evolution operator is purely real, restricting the generalized phase factors to be O(N) matrices for a model in which N-component fermions propagate on links and to the values ±1 for $N=1$, the case we treat. In addition to the link phases, it is useful also to consider the gauge-invariant total phase, modulo 2π, accumulated on passing around each elementary plaquette. In the disorder-free reference system of Fig. 2, this phase has the value π for every plaquette. Randomness can be characterized by the positions of flux lines which thread a subset of plaquettes, changing the value of the accumulated phase on these to 0. Local disorder in link phases generates these flux lines in pairs.

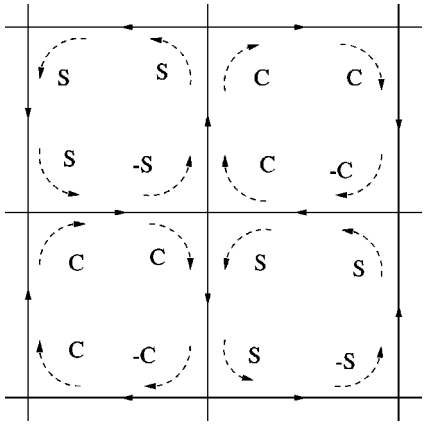

FIG. 2. The network model. Values of the scattering matrix elements, $\pm\cos(\alpha)$ and $\pm\sin(\alpha)$, at nodes on each sublattice are indicated for $p=0$ schematically by $\pm C$ and $\pm S$.

The models we study and the important distinctions between them are as follows. The CF model has flux-line pairs introduced at nodes with probability p, in such a way that for every pair the position vector representing the separation of the two flux lines is oriented in the [1,1] direction on the lattice of Fig. 2. In consequence, both members of a flux-line pair pass through plaquettes with the same circulation, but different pairs may belong to plaquettes with opposite circulation. The RBIM similarly has a pair of flux lines introduced at each node with probability p, but with the difference that all flux lines thread plaquettes of the same circulation.[13,15] Finally, the O(1) model has link phase factors chosen negative with probability p and positive with probability $1-p$. As a result, a pair of flux lines is introduced with probability p into the two plaquettes either side of each link: these plaquettes have opposite circulation. Each of the models is invariant under the transformation $p\rightarrow1-p$, and so we consider only $0\leqslant p\leqslant1/2$. The CF and O(1) models are both (statistically) self-dual for all p under a Kramers-Wannier transformation that takes α to $\pi/2-\alpha$, leaving the line $\alpha=\pi/4$ invariant. The RBIM is not self-dual, except at $p=0$. Finally, the CF and O(1) models are equivalent, under a gauge transformation, on the line $p=1/2$.

Some of the differences in the behavior of these three models can be illustrated by solving their one-dimensional versions, which consist of a single chain of square plaquettes joined by nodes at opposite corners as on a diagonal of Fig. 2. For the RBIM in one dimension, disorder in the sign of the nearest-neighbor exchange interaction can be removed by gauge transformation, and the inverse localization length has

the disorder-independent value $\xi_{\mathrm{RBIM}}^{-1} \equiv \operatorname{arctanh}[|\sin(\alpha)|]$, finite for all $\alpha \neq 0, \pi/2$. For the CF model, an elementary calculation gives

$$\xi_{\mathrm{CF}}^{-1} = |1-2p| \xi_{\mathrm{RBIM}}^{-1}, \tag{1}$$

so that ξ_{CF} diverges as $p \to 1/2$ but is otherwise finite. For the O(1) model we find $\xi_{\mathrm{O(1)}}^{-1} = 0$ for all $p \neq 0,1$. Thus, the localization properties of the one-dimensional CF model at $p \neq 1/2$ are like those of models belonging to the Wigner-Dyson universality classes, in that states are localized, while the absence of localization in the O(1) model mirrors that found previously in quasi-one-dimensional class D systems.[5]

A second useful approach illustrating differences between these models is to consider their two-dimensional versions in the limit of weak interplaquette tunneling ($\alpha \ll 1$ or $\pi/2 - \alpha \ll 1$) and weak disorder ($p \ll 1$).[17] The eigenvalues of the evolution operator U lie on the unit circle and may be written as $e^{-i\epsilon}$. The real ϵ ($-\pi < \epsilon \leqslant \pi$) play the role of excitation energy eigenvalues, and are distributed symmetrically in pairs around $\epsilon = 0$ because U is a real orthogonal matrix. Long-time properties are determined by the part of the spectrum near $\epsilon = 0$, on which we now focus. At zero tunneling, it is sufficient to examine an isolated plaquette. In our disorder-free reference system, the evolution operator for a single plaquette satisfies $U^4 = -1$, and hence $\epsilon = \pm \pi/4$, $\pm 3\pi/4$. For a single plaquette with a flux line added, $U^4 = 1$ and $\epsilon = 0, \pi, \pm \pi/2$. Turning on weak tunneling, it is clear that the spectrum near $\epsilon = 0$ for a large system will arise by hybridization of the $\epsilon = 0$ states from plaquettes with flux lines. In both the RBIM and CF models, there are two scales for this hybridization, because flux lines appear in the system in adjacent pairs associated with plaquettes of the same circulation. The first consequence of tunneling is to remove the degeneracy within each pair, yielding approximate eigenvalues $\epsilon = \pm \epsilon_0$. At small p, pairs are dilute and tunneling between different pairs is not sufficient to generate extended states at $\epsilon = 0$. By contrast, for the O(1) model in this regime there is only one scale for hybridization, since single flux lines appear independently on the set of weakly coupled plaquettes. As a result, metallic behavior is not excluded even at $p, \alpha \ll 1$.

Our results from numerical simulation supplement this qualitative discussion. We study the CF and O(1) models in cylindrical geometry via the transfer matrix T, obtaining the positive Lyapunov exponents $0 \leqslant \nu_1 \leqslant \cdots \leqslant \nu_M$ in a system of width $M' = 2M$ links. These exponents define inverse correlation lengths, and from their dependence on M' we identify the phases present in each two-dimensional model, using established scaling ideas.[19,20] A crucial technical aspect of these calculations is our discovery that the standard algorithm[19,20] has a serious instability to round-off errors throughout much of the phase diagram of both models. More specifically, we find that the smallest positive Lyapunov exponent ν_1 may be either identically zero or exceptionally small ($\nu_1 \ll M^{-1}$). [The first happens in the O(1) model for all p and α and in the CF model on the self-dual line $\alpha = \pi/4$; the second happens in the metallic phase of the CF model.] Under these circumstances, numerical noise from round-off errors generates a systematic positive error in the value obtain for ν_1. From an analytical theory[18] of the instability, we find that the error in ν_1 decreases with reduced noise amplitude η only as $|\ln(\eta)|^{-1}$. This instability can be cured by making explicit use in numerical calculations of the structure imposed on T by current conservation and the symmetry of class D, as described elsewhere.[18] The results we obtain in this way for the CF model differ significantly from those of Ref. 11.

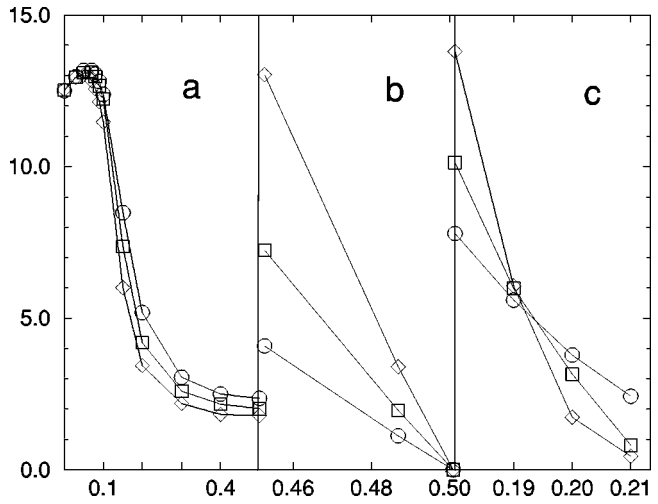

FIG. 3. Behavior of the CF model in systems of width $M' = 64$ (○), 128 (□), and 256 (◇). (a) The self-dual line: $M'\nu_2$ as a function of p. (b) Quantum-Hall-type transition: $M'\nu_1$ as a function of $\sin^2(\alpha)$ at $p = 0.1$. (c) Insulator-metal transition: $M'\nu_1$ as a function of p at $\sin^2(\alpha) = 0.19$.

Evidence in support of the phase diagram of Fig. 1 for the CF model is presented in Fig. 3. First we consider behavior as a function of p on the self-dual line $\alpha = \pi/4$. On this line we believe that ν_1 is identically zero [as in the one-dimensional O(1) model]. For example, at $p = 1/2$ and $\alpha = \pi/4$ we obtain in systems of length $L = 5 \times 10^5$ the bounds $\nu_1 < 1.5 \times 10^{-3}$ at width $M' = 4$ and $\nu_1 < 1.5 \times 10^{-4}$ at $M' = 256$. In order to search for a possible multicritical point on the self-dual line, we therefore examine the value of ν_2.[21] If there is a multicritical point at $p = p_{\mathrm{MC}}$, one expects the amplitude ratio $M'\nu_2$ to show three regimes at large M', as a function of p. For $p < p_{\mathrm{MC}}$, the scaling flow is towards smaller p and $M'\nu_2$ has a p-independent value governed by the critical point at $p = 0$. At $p = p_{\mathrm{MC}}$, a distinct limiting value arises from the multicritical point. And for $p > p_{\mathrm{MC}}$, the scaling flow of the conductivity in the metallic phase towards larger values means that $M'\nu_2$ will slowly decrease towards zero with increasing M'. The data shown in Fig. 3(a) are consistent with this scenario, although the position of the multicritical point is not well determined: we find the bounds $0.05 \leqslant p_{\mathrm{MC}} \leqslant 0.15$. Second, we consider behavior as a function of α at fixed $p < p_{\mathrm{MC}}$. A quantum-Hall-type transition is observed on crossing the self-dual line, as illustrated in Fig. 3(b): $M'\nu_1$ increases with M' for $\alpha \neq \pi/4$ (localization) and vanishes as $\alpha \to \pi/4$ (delocalization). This transition is expected[7] to be in the universality class of the pure Ising transition, because the disorder strength scales towards

zero, as in the RBIM at small p. Third, we examine behavior as a function of p for $\alpha \neq \pi/4$, shown in Fig. 3(c). Phases can be identified using the variation of $M'\nu_1$ with M'. At small p, $M'\nu_1$ increases with M', as expected for an insulator, and at larger p, $M'\nu_1$ decreases rapidly with increasing M', indicating a thermal metal. The critical point $p_C(\alpha)$ is identified by the crossing of curves for different M'. In this way, we arrive at the phase diagram for the CF model displayed in Fig. 1.

We believe that the O(1) model has only a metallic phase and has ν_1 identically zero for all $p \neq 0$. Our calculations cover the range $0.1 \leqslant p < 0.5$ and $0.1 \leqslant \sin^2(\alpha) \leqslant 0.5$. If the model were to support a localized phase, it should appear at small p, α. As an illustration of the absence of localization, at $p = 0.1$, $\sin^2(\alpha) = 0.1$, we calculate for $M' = 16$, $\nu_1 < 10^{-3}$ in the O(1) model, while $\nu_1 = 0.83$ in the CF model.

In summary, we find that two-dimensional models for localization in the symmetry class D can have quite different behavior according to the form of disorder. Several additional points deserve emphasis. The metallic phase of the CF model is self-dual, as is its multicritical point. By contrast, the RBIM is not self-dual but has higher supersymmetry at its multicritical point.[13] There is little reason to suppose that these two multicritical points are in the same universality class. Separately, the apparent absence of an insulating phase in the O(1) model is remarkable, because the bare conductivity becomes small when $\alpha \to 0$ or $\pi/2$. Recently, it has been emphasized that the target manifold of the class D nonlinear σ model is not connected,[9] and this means that domain wall excitations can occur in the σ model, which must be described by additional parameters and have not been taken into account in weak-coupling analyses so far. It is likely that these domain walls in the σ model language are connected with the richness of phases in this symmetry class. In that context, the O(1) model with $p = 1/2$ is known to be a special case, since it maps to a σ model without domain walls: this fact suggests that proliferation of domain walls may be necessary for localization.[9,13,15]

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[21] A complication arises from the fact that the CF model is not (statistically) invariant under 90° rotations, except at $p = 0$ and $p = 1/2$. Because of this (p- and α-dependent) anisotropy, the ν_n have been calculated throughout as the geometric mean of the Lyapunov exponents for cylinders with perpendicular orientations relative to the lattice.