Disorder Induced Transitions in Layered Coulomb Gases and Superconductors

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A 3D layered system of charges with logarithmic interaction parallel to the layers and random dipoles is studied via a novel variational method and an energy rationale which reproduces the known phase diagram for a single layer. Increasing interlayer coupling leads to successive transitions in which charge rods correlated in $N > 1$ neighboring layers are nucleated by weaker disorder. For layered superconductors in the limit of only magnetic interlayer coupling, the method predicts and locates a disorder induced defect-unbinding transition in the flux lattice. While $N = 1$ charges dominate there, $N > 1$ disorder induced defect rods are predicted for multilayer superconductors.

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Topological phase transitions induced by quenched disorder are relevant for numerous physical systems. Such transitions are likely to shape the phase diagram of type II superconductors. It was proposed [1] that the flux lattice (FL) remains a topologically ordered Bragg glass at low field, unstable to proliferation of dislocations above a threshold disorder or field, providing one scenario for the controversial "second peak" line [2,3]. Another scenario [4] is based on a disorder induced decoupling transition (DT) responsible for a sharp drop in the FL tilt modulus. Furthermore, for the *pure* system, it was shown recently [5] that, in the absence of interlayer Josephson coupling, point "pancake" vortices, i.e., defects such as vacancies and interstitials in the FL, are nucleated at a temperature $T_{\rm def}$, distinct from melting above some field. It is believed that this pure system topological transition merges with the thermal DT [6,7] once the Josephson coupling is finite, being two anisotropic limits of the same transition [8]. In this combined DT-defect transition superconducting order is destroyed while FL positional correlations are maintained. Thus an interesting possibility is that a similar, but now disorder induced, vacancy-interstitial unbinding transition can be demonstrated in 3D layered superconductors, relevant to many layered and multilayer materials [2,9].

In 2D recent progress was made to describe disorder induced topological transitions, in terms of Coulomb gases of charges with logarithmic long range interactions. It was shown [10–13] that quenched random dipoles lead to a transition, via defect proliferation, at a finite threshold disorder, even at $T = 0$. The charges then see a logarithmically correlated random potential, a unique type which allows a nontrivial phase transition.

In this Letter we develop a theory for a 3D defect-unbinding transition in the presence of disorder. It is achieved for systems which can be mapped onto a layered Coulomb gas with quenched random dipoles, in which the interaction energy between two charges on layers n and n' is $2J_{n-n'}\ln r$, with r the charge separation parallel to the layers. This system is realized by the FL in layered superconductors [2,9,14] with only interlayer magnetic coupling, for which we predict and locate the vacancy-interstitial unbinding transition. Indeed, as we show, disorder induced deformations of the lattice result in random dipoles as seen by the defects. Our results correspond to systems with negligible Josephson coupling, e.g., multilayer systems; with finite Josephson coupling the phase transition found here is a lower bound on the combined DT-defect transition field.

To study this problem, we develop an efficient variational method which allows for fugacity distributions, known [13] to be important in 2D as they become broad at low T. We test the method on a single layer and reproduce the phase diagram, known from renormalization group (RG) with a $T = 0$ disorder threshold $\sigma_{\rm cr} = 1/8$ [15]. For the two-layer system, we find that above a critical anisotropy $\eta \equiv -J_1/J_0 = \eta_c = 1 - 1/\sqrt{2}$ the single layer-type transition is preempted by a transition induced by bound states of two pancake vortices on the two layers with $\sigma_{\rm cr} < 1/8$. We develop a $T = 0$ energy rationale by an approximate mapping to a Cayley tree problem and find that it reproduces the two-layer result. Extension to many layers with only nearest layer coupling shows a cascade of transitions in which the number of correlated charges on N neighboring layers increases, while the critical disorder decreases with η, with $N \to \infty$, $\sigma_{\rm cr} \to 0$ as $\eta \to 1/2$. For layered superconductors, we expect that the $N = 1$ state dominates and find its phase diagram. Varying the system parameters by forming multilayers allows for realization of the new $N > 1$ phases.

We study the Hamiltonian:

$$\mathcal{H} = -\tfrac{1}{2}\sum_{\mathbf{r}\neq\mathbf{r}'}\sum_{n,n'} 2J_{n-n'}s_n(\mathbf{r})\ln(\mathbf{r}-\mathbf{r}')s_{n'}(\mathbf{r}')\,, \quad (1)$$

$$-\sum_{\mathbf{r},n} V_n(\mathbf{r})s_n(\mathbf{r})\,, \quad (2)$$

where $s_n(\mathbf{r}) = \pm 1, 0$ define the positions $\mathbf{r}$ of charges on the nth layer, and $V_n(\mathbf{r})$ is a disorder potential with long range correlations $\overline{V_n(\mathbf{q})V_{n'}(-\mathbf{q})} = 4\pi\sigma J_0^2\Delta_{n-n'}/q^2$ with $\Delta_0 = 1$ (the short distance cutoff being set to unity). For simplicity we start with uncorrelated disorder from layer to layer $\Delta_{n-n'} = \delta_{nn'}$ with

$$\overline{[V_n(\mathbf{r}) - V_n(\mathbf{r}')]^2} = 4\sigma J_0^2 \ln|\mathbf{r} - \mathbf{r}'| \tag{3}$$

representing quenched dipoles on each layer. We develop first a simple energy rationale for $T = 0$. For a single layer it corresponds to either using [10,17] a "random energy model" approximation [18] or, more accurately, to a representation in terms of directed polymers on a Cayley tree (DPCT) [12] shown to emerge [13] (as a continuum branching process) from the Coulomb gas RG of the single layer problem. Schematically, the tree has independent random potentials (Fig. 1) v_i on each bond with variance $\overline{v_i^2} = 2\sigma J_0^2$. After l generations, one has $\sim e^{2l}$ sites which are mapped onto a 2D layer; i.e., two points separated by $r \sim e^l$ have a common ancestor at the previous $l \approx \ln r$ generation. Each point $\mathbf{r}$ has a unique path on the tree with $v_1, \ldots, v_l$ potentials and is assigned a potential $V(\mathbf{r}) = v_1 + \cdots + v_l$. Since all bonds previous to the common ancestor are identical $\overline{[V(\mathbf{r}) - V(\mathbf{r}')]^2} = 2\sum_{i=1}^{l} \overline{v_i^2}$ reproducing (3) on each layer. Exact solution of the DPCT [19] yields the energy gained from disorder $V_{\min} = \min_{\mathbf{r}} V(\mathbf{r}) \approx -\sqrt{8\sigma}\, J_0 \ln L$ for a volume L^2, with only $O(1)$ fluctuations [13], i.e., $-\sqrt{8\sigma}\, J_0$ per generation with $l = \ln L$.

Optimal energy configurations for M coupled layers are constructed considering N neighboring layers with a $+,-$ pair on each layer and no charges on the other layers. We can take $J_0 > 0$ and $J_{n\neq 0} \leq 0$ so that *equal* charges on different layers attract. The DPCT representation now involves, on a single tree, N $+$ polymers (each seeing different disorder) and N $-$ polymers. A plausible configuration is that the $+$ charges bind within a scale L^ϵ ($0 \leq \epsilon \leq 1$); so do the $-$ charges, while the $+$ to $-$ charge separations define the scale L. Its tree representation (Fig. 1) has $2N$ branches with $\epsilon \ln L$ generations, i.e., an optimal energy of $-2N\sqrt{8\sigma}\, J_0 \epsilon \ln L$. On the scale between L^ϵ and L, the $+$ charges act as a single charge with a potential $\sum_{n=1}^{N} V_n(\mathbf{r})$ of variance $N\sigma$, hence, the optimal energy is $-2\sqrt{8N\sigma}\, J_0(1 - \epsilon)\ln L$. The total disorder energy is [20]

$$E_{\text{dis}} \approx -2J_0\sqrt{8\sigma}\,[\epsilon N + (1 - \epsilon)\sqrt{N}\,]\ln L\,. \tag{4}$$

The competing interaction energy E_{int} is the sum of the one for the $+-$ pairs, $[2J_0 N + 4\sum_{n=1}^{N} J_n(N - n)]\ln L$ and for the $++\ /\ --$ pairs, $-4\sum_{n=1}^{N} J_n(N - n)\epsilon \ln L$. The total energy $E_{\text{tot}} = E_{\text{dis}} + E_{\text{int}}$ being linear in ϵ, its minimum is at either $\epsilon = 1$ or $\epsilon = 0$. Since $\epsilon = 1$ implies that the $+$ charges unbind, it is sufficient to consider $\epsilon = 0$ with all $N \geq 1$, i.e., a rod with N correlated charges has energy (with $\eta_n = -J_n/J_0$)

$$E_{\text{tot}} = 2J_0 N\left[1 - 2\sum_{n=1}^{N} \eta_n\left(1 - \frac{n}{N}\right) - \sqrt{\frac{8\sigma}{N}}\,\right]\ln L\,. \tag{5}$$

Disorder induces the N vortex state at the critical value:

$$\sigma_{\text{cr}} = \frac{N}{8}\left[1 - 2\sum_{n=1}^{N} \eta_n\left(1 - \frac{n}{N}\right)\right]^2 \tag{6}$$

(i.e., $E_{\text{tot}} = 0$). Consider first only nearest neighbor coupling $\eta_l = \eta_1 \delta_{l1}$. Then σ_{cr} is minimal at $N = 1$ with $\sigma_{\text{cr}} = 1/8$ if $\eta_1 < 1 - 1/\sqrt{2}$. For larger anisotropies successive N states form at $1/(1 - 2\eta_1) = 1 + \sqrt{N(N - 1)} \sim N$ with diverging N as $\eta_1 \to \frac{1}{2}$ (Fig. 1) [21].

Consider now J_n of range n_0 constrained by $\sum_n J_n = 0$ as relevant to superconductors; e.g., $\eta_n = \eta_1 e^{-(n-1)/n_0}$ for which $\sigma_{\text{cr}} = (1 - e^{-N/n_0})/8N(1 - e^{-1/n_0})$. For $n_0 \gg 1$, each $\eta_{n\neq 0}$ is small: for $N \lesssim n_0$, the lowest σ_{cr} is at $N = 1$. However, the combined strength of $N \approx n_0$ vortices being significant σ_{cr} has a maximum and decreases back to zero for $N > n_0$ as $\sigma_{\text{cr}} \approx n_0^2/8N$. Hence, $\sigma_{\text{cr}} \to 0$ as $N \to \infty$ and any small disorder seems to nucleate such vortices. The constraint $\sum_n J_n = 0$ manifests that an infinite vortex line has a vanishing $\ln r$ interaction; hence, a logarithmically correlated disorder is always dominant.

The realization of the large N rods depends, however, on the type of thermodynamic limit. Adding to (5) the core energy $E_c N$ and minimizing yields a N-vortex scale

$$L \approx \exp\{E_c\sqrt{N}/[2J_0(\sqrt{8\sigma} - \sqrt{8\sigma_{\text{cr}}}\,)]\}\,. \tag{7}$$

Hence, as $\sigma \to 0$ such states are achievable only when L/N diverges exponentially. Using $\sigma_{\text{cr}} \approx n_0^2/8N$, for $N > n_0^2/8\sigma$ the lowest scale L in this range is achieved at $N = n_0^2/2\sigma$ and leads to a lower bound $L_{\min} \approx \exp[E_c n_0/4J_0\sigma]$ for observing large N states with a given $\sigma < \frac{1}{8}$. For layered superconductors $E_c/J_0 \gg 1$ [22] and $n_0 \gg 1$ and this large N instability occurs at unattainable scales, thus $N = 1$ dominates. One needs $n_0 \approx 2$–3, as in multilayers, to realize the $N > 1$ states.

To substantiate these results, we develop a variational method for M layers which allows for fugacity distributions, an essential feature in the one-layer problem. Disorder averaging (2) in Fourier using replicas yields

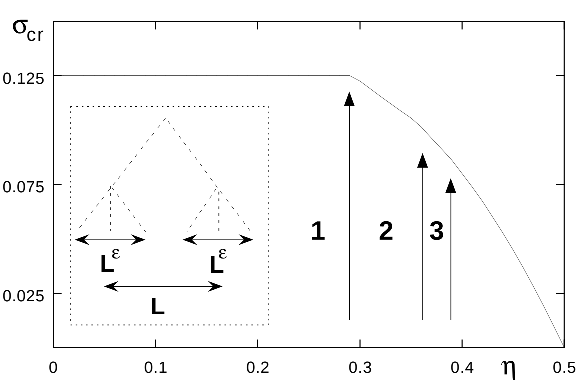

FIG. 1. Critical disorder values with only nearest neighbor coupling J_1 vs the anisotropy $\eta = -J_1/J_0$. Transitions between different N phases are marked with arrows. Inset: the Cayley tree representation (for $N = 3$ neighboring layers) with $+$ charges (at the tree end points) separated by L^ϵ along the layers, and separated by L from the $N = 3$ $-$ charges.

$$\beta \mathcal{H}_r = \frac{1}{2d^2} \int_k \int_q s_a(\mathbf{q}, k)\, (G_0)_{ab}(\mathbf{q}, k) s_b^*(\mathbf{q}, k) + \beta E_c \sum_{\mathbf{r},n} s_{na}^2(\mathbf{r}), \tag{8}$$

where $\beta = 1/T$, $(G_0)_{ab}(\mathbf{q}, k) = (4\pi/q^2)[\beta J(k)\delta_{ab} - \sigma J_0^2 \beta^2 \Delta(k)]$, d is the interlayer spacing [23], $a, b = 1, \ldots, m$ are replica indices, and $m \to 0$ is to be carefully taken. In transforming to a sine-Gordon Hamiltonian [8], it is crucial to keep *all* charge fugacities [13], which yields

$$\beta \mathcal{H}_{\rm SG} = \frac{1}{2} \int_{kq} \chi_a(\mathbf{q}, k)\, (G_0)^{-1}_{ab} \chi_b^*(\mathbf{q}, k) - \sum_{\mathbf{r}} \sum_{\mathbf{s} \neq \mathbf{0}} Y[\mathbf{s}] \exp[i\mathbf{s} \cdot \boldsymbol{\chi}(\mathbf{r})]. \tag{9}$$

From now on, $\mathbf{s} = \{s_{na}\}_{n=1,\ldots,M; a=1,\ldots,m}$ is an integer vector both in layer label and replica space (i.e., of length $M \cdot m$) of entries $0, \pm 1$, and the summation is over all such non-null vectors [also $\chi(\mathbf{r}) \equiv \{\chi_{n,a}(\mathbf{r})\}$, $\mathbf{s} \cdot \boldsymbol{\chi} = \sum_{na} s_{na} \chi_{na}$]. We now look for the best Gaussian approximation of (9) with propagator $G^{-1}_{ab}(\mathbf{q}, k) = (G_0)^{-1}_{ab}(\mathbf{q}, k) + \sigma_c(k)\delta_{ab} + \sigma_0(k)$. The bare fugacity being $Y[\mathbf{s}] = \exp(-\beta E_c \sum_{n,a} s_{n,a}^2)$, the naive approach would be to restrict to charges $\mathbf{s}$ with a single nonzero entry, leading to a uniform fugacity term $\exp(-\beta E_c) \times \sum_{\mathbf{r},n,a} \cos[\boldsymbol{\chi}_{na}(\mathbf{r})]$ and a diagonal k-independent replica mass term. Instead we keep *all* composite charges $\mathbf{s}$, which allow for variational solutions with off-diagonal and k-dependent replica mass terms. This corresponds, respectively, to fluctuations of fugacity and $N > 1$ charge rods being generated and becoming relevant as also seen from RG.

Schematically, we evaluate averages $\langle \cdots \rangle_0$ using a Boltzmann weight with $\beta \mathcal{H}_0 = \frac{1}{2} \int_{\mathbf{q},k} \chi_a(\mathbf{q}, k) G_{ab}(\mathbf{q}, k) \times \chi_b^*(\mathbf{q}, k)$. Hence, the last term of (9) has an average $F[\mathbf{s}] \equiv Y[\mathbf{s}] \langle \exp i\mathbf{s} \cdot \boldsymbol{\chi}(\mathbf{r}) \rangle_0$ which can be written in terms of $\sum_q G_{ab}(q, k) = G_c(k)\delta_{ab} - A(k)$ with $G_c(k) \sim A(k) \sim \ln[1/\sigma_c(k)]$ as $\sigma_c(k) \to 0$. While details of the method are given elsewhere [16], a key element involves rewriting the off-diagonal terms with $A(k)$ as an average over M random Gaussian fugacities w_k:

$$\exp\left\{ \frac{1}{2} \left| \sum_a s_a(k) \right|^2 A(k) \right\} = \left\langle \exp w_k \sum_a s_a(k) \right\rangle_w, \tag{10}$$

where [23] $\langle \cdots \rangle_w = \prod_k \int \cdots e^{-|w_k|^2/2A(k)}\, d^2 w_k / \sqrt{2\pi A(k)}$. This allows one to perform the exact sum on replicas yielding a form $\sum_{\mathbf{s}} F[\mathbf{s}] = \langle Z^m \rangle_w$. The variational equation for $m \to 0$ is $\sigma_c(k) = \langle (\partial^2 \ln Z)/(\partial w_k \partial w_k^*) \rangle_w$ expressed in terms of the fugacity distributions w_k. For a single layer $k = 0$ and Z is a trinomial in terms of exponentials involving $G_c(0)$ and $w_{k=0}$. The solution for the critical line where $\sigma_c(0) \to 0$ is shown in Fig. 2 ($N = 1$ line) reproducing precisely recent RG results. The variational scheme, allowing for all replica charges $\mathbf{s}$, therefore treats disorder correctly. For two layers $kd = 0, \pi$ we need two fugacity distributions w_0, w_π and Z is a "ninomial," i.e., $Z = 1+$ eight exponentials involving $G_c(0)$, $G_c(\pi)$,

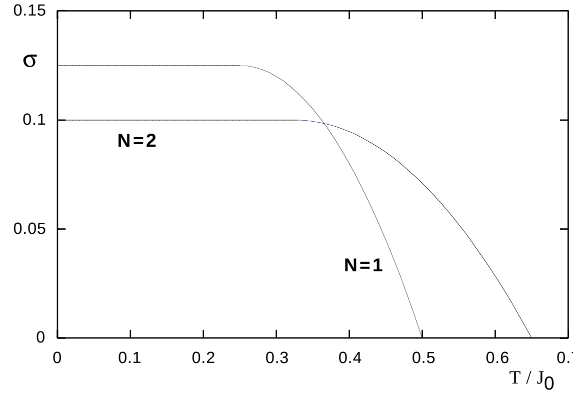

FIG. 2. Phase diagram for the onset of the $N = 1, 2$ instabilities for anisotropy $\eta = 0.35$. At low T two distinct transitions are possible, the first being to the rod $N = 2$ phase. At high T the independent layer $N = 1$ transition dominates.

w_0, w_π. Focusing on the low T boundary, where $\sigma_c(\pi) \sim [\sigma_c(0)]^\alpha \to 0$, we find [16] either (i) $\alpha = 1$ for $\eta_1 < \eta_c = 1 - 1/\sqrt{2}$, representing decoupled layers, or (ii) $\alpha \to \infty$ for $\eta_1 > \eta_c$, representing a $++$ bound state on the two layers. The $T = 0$ energy rationale is therefore reproduced. The phase diagram for two layers with $\eta_c < \eta < 1/2$ is shown in Fig. 2 [15].

For any number of layers, one obtains a simple N rod solution by restricting the sum over $\mathbf{s}$ in (9) to a subclass of charges of the form $s_{na} = s_a \sum_{j=1,N} \delta_{n,n'+j-1}$. The variational solution, of the form $\sigma_c(k) = \sigma_c \phi_N(k)$, reduces to an effective one-layer problem, in term of the structure factor of the rod $s_a(k) s_b^*(k) = \phi_N(k) \equiv \sin^2(Nkd/2)/\sin^2(kd/2)$. Allowing now interlayer disorder correlation via $\Delta(k)$, the N rod becomes critical at

$$\sigma_{\rm cr} = \left(\int_k \phi_N(k) J(k) \right)^2 \Big/ \left(8 J_0^2 \int_k \phi_N(k) \Delta(k) \right). \tag{11}$$

This formula generalizes (6) and can equivalently be obtained within the Cayley tree rationale.

As a direct application, we consider a flux lattice in a layered superconductor with no Josephson coupling and a magnetic field B perpendicular to the layers. We first consider the clean system, reproduce [5] by a systematic derivation of the defect interaction, and then allow for disorder. The FL is composed of pancake vortices displaced from the pth line position $\mathbf{R}_p$ at the nth layer into $\mathbf{R}_p + \mathbf{u}_p^n$. In addition to the pancake vortices composing the FL, we allow for vacancies and interstitials, i.e., defects $s_n(\mathbf{r}) = 0, \pm 1$. These defects couple to the lattice via $\mathcal{H}_{\rm vac} = \sum_{\mathbf{r},p,n,n'} s_{n'}(\mathbf{r}) G_v(\mathbf{R}_p + \mathbf{u}_p^n - \mathbf{r}, n' - n)$, where in Fourier [8] $G_v(\mathbf{q}, k) = (\phi_0^2 d^2 / 4\pi \lambda_{ab}^2 q^2)/[1 + f(\mathbf{q}, k)]$ and $f(\mathbf{q}, k) = (d/4\lambda_{ab}^2 q) \times \sinh qd/[\sinh^2(qd/2) + \sin^2(kd/2)]$; $\phi_0 = Ba^2$ is the flux quantum, a is the FL spacing, λ_{ab} is the penetration length along the layers. To zeroth order in $\mathbf{u}_p^n$ the defects feel a periodic potential fixing their position in a unit cell, hence, $s(\mathbf{q}, k)$ involve only $|q| < 1/a$.

In the limit $q \to 0$ the longitudinal modes, to which defects couple, have for (tilt) elastic energy [24] $\mathcal{H}_{el} = 1/(2d^2a^4)\int_{kq} D(k)\,|\mathbf{u}_L(\mathbf{q},k)|^2$ with $D(k) = \frac{1}{2}\sum_{\mathbf{Q}\neq 0}\times [G_v(\mathbf{Q},k) - G_v(\mathbf{Q},0)] + G_v(k)$, where $\mathbf{Q}$ are reciprocal wave vectors of the lattice and $G_v(k) = \lim_{q\to 0} G_v(\mathbf{q},k)q^2 = \phi_0^2 d^2 k_z^2/[4\pi(1+\lambda_{ab}^2 k_z^2)]$ and $k_z = (2/d)\sin(kd/2)$. The sum on $\mathbf{Q}$ is due to the high momentum components of the magnetic field and is responsible for the nonperfect screening of the defect interaction and to a finite $T_{\rm def}$. Minimizing $\mathcal{H}_{\rm vac} + \mathcal{H}_{el}$ yields then $\mathbf{u}_{\rm vac}(\mathbf{q},k) = i\mathbf{q}s(\mathbf{q},k)G_v(k)a^2/D(k)q^2$. The effective interaction between defects involves the direct one $G_v(q,k)$ and the FL screening which is $\mathcal{H}_{\rm vac} + \mathcal{H}_{el}$ at its minimum. The total effective interaction then has form (8) with

$$J(k) = G_v(k)[1 - G_v(k)/D(k)]/4\pi\,. \qquad (12)$$

Thus the long range interaction is $\sim \ln r$ and its coefficient determines $T_{\rm def} = \frac{1}{2}\int_k J(k) = \frac{1}{2}J_0$. Since $\int_k G_v(k) \sim \phi_0^2 d/\lambda_{ab}^2$, the scale of the melting transition [14], the defect transition occurs before melting and can thus be consistently described only if $D(k) - G_v(k) \ll D(k)$. This is possible if either $d \ll a \ll \lambda_{ab}$, where $J(k) = d\tau' \ln(1 + a^2k_z^2/4\pi)$ with $\tau' = \phi_0^2 da^2/(128\pi^3\lambda_{ab}^4)$ and [5] $T_{\rm def} = \tau' \ln(a/d)$, or for $d > a$, where $J(k) = (d^4/a)\tau' k_z^2 e^{-2\pi d/a}$ leading to $T_{\rm def} = 4(d/a)\tau' e^{-2\pi d/a}$. Remarkably, $D(k) - G_v(k) \ll D(k)$ also yields that the long range response $\mathbf{u}_{\rm vac}(\mathbf{r}) \sim a^2\mathbf{r}/r^2$ to a vacancy at $\mathbf{r} = 0$ is confined to the same layer.

Point disorder deforms the flux lattice, producing quenched dipoles coupling to our defects. Expansion of the disorder energy, valid below the Larkin length [1], and minimization together with $\mathcal{H}_{\rm vac} + \mathcal{H}_{el}$ yields readily (2). A more general argument, valid at all scales, treats $u_{\rm vac}$ as a small perturbation around the Bragg glass configuration. Systematic expansion of the free energy $F = F_{\rm BG} + \frac{1}{d^2a^2}\int_{qk} s(q,k)G_v(q,k)i\mathbf{q}\cdot\langle\mathbf{u}(q,k)\rangle_{s=0} + O(s^2)$ in defect density in a given disorder configuration with $\langle\cdots\rangle$ a thermal average. This shows that disorder induced displacements generate random dipoles $i\mathbf{q}\cdot\langle\mathbf{u}(q,k)\rangle_{s=0}$ for the charges $s(q,k)$. Hence, the defects feel a logarithmically correlated random potential $V_n(\mathbf{r})$ as in (2) and (8) with $\sigma J_0^2\Delta(k) = G_v(k)^2 \lim_{q\to 0} C_{\rm BG}(q,k)/4\pi d^2a^4$, where $C_{\rm BG}(\mathbf{r},n) = \overline{\langle u_L^0(\mathbf{0})\rangle\langle u_L^n(\mathbf{r})\rangle}$ is the disorder averaged correlation in the unperturbed Bragg glass $C_{\rm BG}(q=0,k) \sim 1/[c_{44}^2(k^4 + R_c^{-1}k^3)]$, R_c a Larkin length along c [1]. It yields a k-independent $\Delta(k)$ for $k > 1/R_c$ while $\Delta(k) \sim k$ for $k < 1/R_c$.

Applications to FL depends on the interlayer form of (12) of range $n_0 \approx a/d$ for large a/d. Remarkably, $g(k=0) = 0$, i.e., perfect screening holds as in 2D [5]. Hence, $\sum_n J_n = 0$ and as n_0 is reduced J_0, J_1 dominate the sum, i.e., $\eta_1 \to \frac{1}{2}$ when $d \gg a$. One finds that η_1 crosses the critical value $1 - 1/\sqrt{2}$ when $d/a \approx 1$, depending weakly on a/λ_{ab}. We thus propose that FL in multilayer superconductors, where $d > a$ can be achieved, can show a rich phase diagram with $N > 1$ phases. In layered superconductors $a/d \approx 10$–100 [2] and the $N = 1$ transition at $\sigma_{\rm cr} = 1/8$ dominates for realistic sizes. The disorder induced decoupling transition, neglecting defects, predicted [7] at $\sigma_{\rm dec} = 2$ is thus above the defect transition (with $B \sim \sigma$) in the $B - T$ plane (similarly thermal decoupling occurs at $T_{\rm dec} = 8T_{\rm def}$ for $d \ll a \ll \lambda$). A natural scenario is again of a single transition at σ_c varying from 2 to $1/8$ as the bare Josephson coupling is reduced, e.g., by increasing d in multilayers.

In conclusion, we developed a variational method and a Cayley tree rationale to study layered Coulomb gases. The results are relevant to flux lattices where we predict that as field or disorder are increased a thermodynamic phase transition will occur, affecting magnetization, transport, and the critical current. We also propose new $N > 1$ phases for $d \gtrsim a$. The present methods may be useful for other 2D disordered systems, such as the quantum Hall.

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[21] If $\eta_1 > \frac{1}{2}$, the defects form a lattice even without disorder.
[22] $E_c/J_0 \approx (\lambda_{ab}/a)^2/\ln(a/d)$ in that case.
[23] $\int_q \equiv \int \frac{d^2q}{(2\pi)^2}$, $\int_k \equiv \frac{1}{Md}\sum_k \to \int_{-\pi/d}^{\pi/d}\frac{dk}{2\pi}$ for large M, $J(k) = d\sum_n J_n e^{ikdn}$, similarly for $\Delta(k)$, $s_a(k)$.
[24] T. R. Goldin and B. Horovitz, Phys. Rev. B **58**, 9524 (1998).