Multiple Andreev and elastic interface scattering in superconductor-normal-metal-superconductor junctions

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Current-voltage characteristics of superconductor-normal-metal-superconductor junctions are calculated. Multiple Andreev reflections and the effect of normal elastic scattering at the superconductor-normal-metal interfaces are considered. The normal reflections of quasiparticles appear due to the mismatches in Fermi velocities and effective masses of superconductors and weak link. The wave-packet solution of nonequilibrium time-dependent Bogoliubov–de Gennes equations was obtained with the help of the boundary conditions on the wave function of the quasiparticle. At voltages of interest for multiple Andreev reflections ($eV<2\Delta$), the result is applied to the rather thick weak links where the density of bound Andreev states is small.

I. INTRODUCTION

Superconductor-normal-metal (SN) contacts and superconductor-normal-metal-superconductor (SNS) junctions exhibit nonlinear electron transport behavior.[1–3] The dynamic properties of long SNS junctions with a high concentration of nonmagnetic impurities (dirty limit) were studied earlier[4] in a model where the transmission probability for a quasiparticle to pass through the SN boundary was one. The proximity effect was shown to be responsible for the enhancing of dynamical conductivity at low voltages. This proximity effect was considered in Ref. 5 as a possible explanation for the zero-bias anomaly in the dirty limit at the interface between superconductor and semiconductor. Non-Ohmic properties may appear because the electrical transport through the SN interface requires the conversion of normal current to supercurrent, a process known as Andreev reflection (AR):[6] at the SN boundary an incident electron from the normal metal with subgap energy $|E|<\Delta$ is reflected into a hole. The missing charge $2e$ is removed as a supercurrent. Subharmonic gap structure was observed in superconducting tunnel junctions[7] as well and this phenomenon was analyzed in terms of multiple Andreev reflections.[8]

The importance of multiple AR's on transport through SN contacts and SNS weak links was explained in Refs. 9–12. In the theory[10–12] the quasiparticles are divided into two subpopulations with nonequilibrium distribution functions that cause their direction of motion. Applying simple energy arguments and boundary conditions at two interfaces Octavio *et al.*[11] and Flensberg, Hansen, and Octavio[12] obtained equations which relate these distribution functions to Andreev and normal reflection probabilities. In another approach, Kümmel, Gunsenheimer, and Nicolsky[13] got an expression for temporally and spatially averaged current density in a relaxation-time approximation. They solved time-dependent Bogoliubov–de Gennes equations (BdGe's) for SNS junctions. The boundary conditions were applied to the wave function which is related to the amplitude of AR (the normal reflections of quasiparticles from SN interfaces were neglected in Ref. 13). For calculating the current through the SNS weak link, the wave-packet solution of the BdGE was used. The expression for the current includes summing over spatially quantized Andreev states which seem to be lacking in the theory of Refs. 11 and 12. Here we clarify this point. Considering mismatches between Fermi velocities and effective masses at the boundaries of superconductors and normal metals, we also find the wave-packet solution of non-equilibrium BdGE's. From this we obtain the current of a pure SNS junction and discuss approximations which are used for calculating the density of quasiparticle states as well as the probability of the tunneling processes.

II. MODEL AND BASIC EQUATIONS

Let us consider a pure SNS system with step pair potential $\Delta(z)$ such that $\Delta(z)=0$ if $|z|<d/2$ (the z axis is taken to be normal to the SN interface, and d is the thickness of the normal-metal layer). An external applied bias voltage exists across the normal metal only and is defined by a time-dependent vector potential $A(zt)=Vct/d$.[13] Thus the time-dependent BdGE's are

$$
\begin{aligned}
i\hbar\frac{\partial u_p}{\partial t}&=\left[p^{+}\frac{1}{2m}p^{+}-\mu\right]u_p+\Delta v_p\ ,\\
i\hbar\frac{\partial v_p}{\partial t}&=-\left[p^{-}\frac{1}{2m}p^{-}-\mu\right]v_p+\Delta u_p\ ,
\end{aligned}
\tag{1}
$$

where u_p,v_p are the electron- and hole-like quasiparticle wave functions,

$$p^{\pm}=-i\hbar\frac{\partial}{\partial z}\pm\frac{e}{c}A(zt)\theta(d/2-|z|)\ .$$

Here $\theta(x)$ is the Heaviside step function which is equal to 1 if $x>0$ or to 0 when $x<0$. Effective masses m and Fermi velocities v are z dependent: m,v are equal to m_s,v_s if $|z|>d/2$, or to m_n,v_n if $|z|<d/2$. We have used the

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homogeneity of the junction in the x,y directions (parallel to the junction) and made a Fourier transformation of the wave functions on these coordinates. Therefore the z-dependent chemical potential μ is equal to $m_n v_n^2 \xi^2/2$ if $|z|<d/2$, or to $m_s v_s^2 \eta^2/2$ when $|z|>d/2$. Here $\xi^2=1-p^2/(m_n v_n)^2$, $\eta^2=1-(1-\xi^2)/(\nu\gamma)^2$, $\nu=m_n$, and $\gamma=v_s/v_n$. The relation between ξ and η is the consequence of continuity at the SN interface of the momentum component parallel to the junction.[14] In the weak link, the solution of Eq. (1) can be written as $\psi_n(t)=\int dE\, \psi_n(Et)$, $\psi_n(Et)=\psi_n^+(Et)+\psi_n^-(Et)$, where ψ_n^+,ψ_n^- are spinors:

$$\psi_n^\pm(Et)=C_1^\pm(E)\begin{bmatrix}1\\0\end{bmatrix}u^\pm(Et)+C_2^\pm(E)\begin{bmatrix}0\\1\end{bmatrix}v^\pm(Et)\ . \qquad (2)$$

The form of the wave functions $u^\pm(Et)$ and $v^\pm(Et)$ follows from the asymptotic representation of Airy functions. The latter are exact solutions of Eq. (1) in the normal region and can be substituted by (2) if the chemical potential $\mu \gg (E\pm V),\Delta$ which is the approximation of our case. So we have

$$\begin{aligned}u^\pm(Et)&=\exp[-i/\hbar(E+eVz/d)t\pm ik(E)z]\ ,\\ v^\pm(Et)&=\exp[-i/\hbar(E-eVz/d)t\pm ik(-E)z]\ ,\end{aligned} \qquad (3)$$

where

$$k(E)=m_n v_n \xi+(E+eVz/2d)/v_n\xi\ .$$

In superconductor banks ($|z|>d/2$) the solution for energies $|E|<\Delta$ has the form

$$\begin{aligned}\psi_s(Et)=e^{-iEt}\Bigg\{&\left[D_1^+e^{ik^+z}\begin{bmatrix}1\\a\end{bmatrix}+D_2^-e^{-ik^-z}\begin{bmatrix}1\\a^{-1}\end{bmatrix}\right]b_R\theta(z-d/2)\\ &+\left[D_1^-e^{-ik^+z}\begin{bmatrix}1\\a\end{bmatrix}+D_2^+e^{ik^-z}\begin{bmatrix}1\\a^{-1}\end{bmatrix}\right]b_L\theta(-z-d/2)\Bigg\}\ ,\end{aligned} \qquad (4)$$

where

$$b_{L,R}=\begin{bmatrix}\exp(i\phi_{L,R}/4)\\ \exp(-i\phi_{L,R}/4)\end{bmatrix}\ ,$$

and

$$k^\pm=m_s v_s\eta\pm i\Omega(\varepsilon)\Delta/(\eta v_s)\ .$$

ϕ_L,ϕ_R are constant phases of the order parameters in the left or right superconductors, respectively, $\Omega(\varepsilon)=\sqrt{1-\varepsilon^2}$, $a=\varepsilon-i\Omega(\varepsilon)$, and the dimensionless energy $\varepsilon=E/\Delta$ is introduced. For $|E|>\Delta$ the solution is obtained by analytic continuation of Eq. (4) (see later). This range of energies will yield a contribution to the current with Andreev reflection probability $A(E)$ which depends upon the energies above the gap. $A(E)$ rapidly decreases when $|E|>\Delta$ and even faster if the elastic reflections are considered.

The spinors $\psi_s(t)=\int dE\,\psi_s(Et)$ and $\psi_n(t)$ obey boundary conditions from which all coefficients $D_{1,2}$ and $C_{1,2}$ can be defined. At NS interfaces $\psi(t)$ and

$$(m)^{-1}\left[\hbar\frac{\partial}{\partial z}+\tau_3\frac{e}{c}A(zt)\theta(d/2-z)\right]\psi(t)$$

(here τ_3 is the Pauli matrix) are continuous. In the stationary limit ($V=0$) this yields the dispersion equation for discrete Andreev levels[14,15] (for the case $v_n=v_s$, $m_n=m_s$, see also Refs. 16–18). In the nonstationary case ($V\neq 0$) we obtain recursive relations between the coefficients $C_{1;2}^\pm(E)$ from different energy intervals,[13] and thus matching the normal-region wave function (2),(3) to the wave function (4) from the superconductor region results in the following equations for $C_{1;2}^\pm(E)$:

$$\begin{aligned}&C_1^+(\varepsilon+u)\gamma^+(\varepsilon+u/2)\exp[-id_0(\varepsilon+u/2)]-iC_1^-(\varepsilon+u)\sqrt{r^2-1}\,\Omega(\varepsilon+u/2)e^{iq(u)}\\ &\quad=e^{0.5i\phi}[C_1^+(\varepsilon-u)\gamma^-(\varepsilon-u/2)\exp[id_0(\varepsilon-u/2)]+iC_1^-(\varepsilon-u)\sqrt{r^2-1}\,\Omega(\varepsilon-u/2)e^{-iq(-u)}]\ ,\\ &iC_1^+(\varepsilon+u)\sqrt{r^2-1}\,\Omega(\varepsilon+u/2)e^{-iq(u)}+C_1^-(\varepsilon+u)\gamma^-(\varepsilon+u/2)\exp[id_0(\varepsilon+u/2)]\\ &\quad=e^{0.5i\phi}[C_1^-(\varepsilon-u)\gamma^+(\varepsilon-u/2)\exp[-id_0(\varepsilon-u/2)]-iC_1^+(\varepsilon-u)\sqrt{r^2-1}\,\Omega(\varepsilon-u/2)e^{iq(-u)}]\ ,\end{aligned} \qquad (5)$$

where

$$\begin{aligned}&\gamma^\pm(\varepsilon)=\varepsilon\pm ir\Omega(\varepsilon)\ ,\quad r=(1+\rho^2)/2\rho\ ,\\ &\rho=\gamma\eta/\xi\ ,\quad q(u)=d(m_n v_n\xi+u\Delta/4v_n\xi)\ ,\\ &u=eV/\Delta\ ,\quad d_0=d\Delta/(\xi v_n)\ ,\end{aligned} \qquad (6)$$

and ϕ is the phase difference between left and right superconductors. The coefficients $C_2^\pm(E)$ can be expressed through $C_1^\pm(E)$ as

$$\begin{aligned}&\exp(i\phi_{L,R}/2)C_2^\pm(\varepsilon-u/2)\\ &\quad=\mp iC_1^\mp(\varepsilon+u/2)(r^2-1)^{0.5}\Omega(\varepsilon)e^{\pm iq(u)}\\ &\qquad+C_1^\pm(\varepsilon+u/2)\gamma^\pm(\varepsilon)\exp(\mp id_0\varepsilon)\ ,\\ &\exp(i\phi_{R,L}/2)C_2^\pm(\varepsilon+u/2)\\ &\quad=C_1^\pm(\varepsilon-u/2)\gamma^\mp(\varepsilon)\exp(\pm id_0\varepsilon)\\ &\qquad+C_1^\mp(\varepsilon-u/2)(r^2-1)^{0.5}\Omega(\varepsilon)e^{\mp iq(-u)}\ .\end{aligned} \qquad (7)$$

When $V \to 0$ (stationary limit) Eqs. (5) are transformed into a system of homogeneous equations for unknown functions $C^{\pm}_{1;2}(E)$; this system has nontrivial ($\neq 0$) solutions if its determinant is zero. This condition gives the dispersion equation for quasidiscrete Andreev levels:[14,15]

$$D(\varepsilon_k)=[r^2-(1+r^2)\varepsilon_k^2]\cos(2\varepsilon_k d_0) -2\varepsilon_k \Omega(\varepsilon_k) r\sin(2\varepsilon_k d_0)+\cos\phi -(r^2-1)(1-\varepsilon_k^2)\cos(2m_n v_n d\xi)=0 . \quad (8)$$

We note that in Eq. (5), as well as in the relations (7), Andreev amplitudes $\gamma^{\pm}$ appear with the factors $\exp(\pm d_0 \varepsilon)$ connected to the coherence length v_n/Δ while the normal reflection has a phase which is changed on the atomic scale $\sim (v_n m_n)^{-1}$. Now we split the energy into intervals of width eV and with the help of Eqs. (5) and (7) relate the coefficients $C^{\pm}_{1;2}(E)$ from the nth interval to the one with $n=0$. Due to (7), it is possible to use one set of coefficients $C^{\pm}(E)$ which represent $C^{\pm}_2(E)$ for odd and $C^{\pm}_1(E)$ for n (here $n>0$):

$$\begin{pmatrix} C^+(\varepsilon \pm un) \\ C^-(\varepsilon \pm un) \end{pmatrix} = \prod_{j=1}^{n} M_{a,b}(j,\varepsilon) \begin{pmatrix} C_1^+(\varepsilon) \\ C_1^-(\varepsilon) \end{pmatrix} ,$$
$$M_{a,b}(j,\varepsilon)=\begin{pmatrix} \gamma^{\mp}(\zeta_\pm)\exp(\pm i d_0 \zeta_\pm) & \pm i\sqrt{r^2-1}\,\Omega(\zeta_\pm)\exp(\pm i S_j^{\pm}) \\ \mp i\sqrt{r^2-1}\,\Omega(\zeta_\pm)\exp(\mp i S_j^{\pm}) & \gamma^{\pm}(\zeta_\pm)\exp(\mp i d_0\zeta_\pm) \end{pmatrix} , \quad (9)$$

where $\zeta_\pm = \varepsilon \pm u(j-\frac{1}{2})$, $S_j^{\pm}=(-1)^j q[\pm(-1)^j u]$, and the upper sign corresponds to M_a, and the lower to M_b. The coefficients $C_1^{\pm}(\varepsilon)$ as in Ref. 13 are chosen freely within an energy interval of width 2 eV as a Gaussian spectral density

$$C_1^{\pm}(\varepsilon)=\frac{N(t)}{\sqrt{\pi\delta^2}}\exp\{-[(E_k \pm eV/2-E)/\delta]^{22}+id_0E/2\} , \quad (10)$$

with $\delta < |2eV|$ and E_k the spectrum of the quasiparticles in an equilibrium SNS junction. When the energy argument of $\gamma^{\pm}(\varepsilon)$ and $\Omega(\varepsilon)$ in Eq. (9) exceeds the gap $\Delta(\varepsilon>1)$, these functions are multiplied by a factor $[D(\varepsilon)]^{-1}$ [see Eq. (8)] (we follow this rule below also). Such a substitution was done in analogy with the equilibrium state. In equilibrium, the voltage $V=0$ and the form of the Andreev amplitude is easily obtained. We took this analogy to generalize relation (9) to the energies above the gap.

The wave-packet solution is the sum over all n,

$$w^{\pm}(kt)=\sum_n w_n^{\pm}(kt) ,$$
$$w_n^{\pm}(kt)=\int_{2un_k}^{2u(1+n_k)} d\varepsilon\, C^{\pm}(\varepsilon+un) w_N^{\pm}(\varepsilon+un) . \quad (11)$$

Here we introduced common notations for the electronlike u and holelike v functions: $w_N^{\pm}(\varepsilon+un)$ is equal to $u^{\pm}$ for even n and to $v^{\pm}$ for odd n [see Eq. (3)]. The superscripts $+(-)$, as before, stand for positive (negative) components of momentum of the quasiparticle.

III. ANDREEV CURRENT IN SNS JUNCTION

In the relaxation-time approximation, the current density has a form similar to that of the equilibrium state,

$$J=-\frac{e}{m_n}\sum_{k,i=\pm}\{f_0(E_k)u^i(kt)^* P u^i(kt) -[1-f_0(E_k)]v^i(kt)Pv^i(kt)^*\}+\text{c.c.} , \quad (12)$$

where $f_0(E_k)$ is the Fermi distribution function, and P is the gauge-invariant momentum operator. Let us consider the action of this operator on functions $u^{\pm}$ and $v^{\pm}$:

$$Pv^{\pm}(kt)=\mp k_h v^{\pm}(kt) , \quad Pu^{\pm}(kt)=\pm k_e u^{\pm}(kt) , \quad (13)$$

where $k_{h,e}=m_n v_n \xi \pm E_k/v_n\xi$ are equilibrium momenta for Andreev reflected electrons and holes. Both Eqs. (13) are true in the approximation when the energy gained by the quasiparticle in the electric field in the junction is much less than the Fermi energy. However, in the $n=0$ term for $u^{\pm}$,

$$Pu^{\pm}_{n=0}(kt)=[k_e \pm eV/(2v_n\xi)]u^{\pm}_{n=0}(kt) ,$$

we also kept the momentum change in the electric field. This gives the normal (Ohmic) contribution to the current. So the $n=0$ term in the current consists of the part which has no Andreev reflections. In this part k_e is exactly canceled and only the electric-field change of momentum remains. We note that the Ohmic contribution represents the whole $n=0$ part of the current only in the absence of elastic reflections from the NS boundary. That is the case which is considered in Ref. 13. Thus the nonlinear part of the Andreev current density takes the form

$$J_{\rm AR}=-\frac{e}{m}\sum_k\{f_0(E_k)k_e-[1-f_0(E_k)]k_h\} \times[|w^+(kt)|^2-|w^-(kt)|^2] . \quad (14)$$

We substitute the sum (11) into this expression. Quadratic terms $|w^{\pm}(kt)|^2$ consist of the double sum over n,n' with factors $\exp[i(z_0-t)(n-n')u]$ and $\exp[i(z_0-t)(n-n'\pm1)u]$, (here $z_0=z\Delta/\xi v_n$, $u=eV/\Delta$). The terms with the second exponent include odd powers of fast-oscillating phase factors [$\exp iq(u)$]; therefore after averages on the atomic scale they become small. Here we keep only diagonal terms with $n=n'$. This is a good approximation if the inequality $ud_0 \gg 1$ holds. When $u<1$, the latter is valid for rather long junctions $d>\xi_0 \sim v_n/\Delta$, but for such long junctions the density of bound Andreev states is small and so they weakly influence $J_{\rm AR}$. The subgap density of states used for numerical calculations in Ref. 13 represents a short junction for which the oscillating terms with $n \neq n'$ cannot be neglected in the range of voltages $u<1$.

The last undefined quantity is the time-dependent normalization constant $N(t)$ [see Eq. (10)]. As in Ref. 13, we normalize the wave-packet solution $w(kt)$ to the probability $P_N(E_k)$ to find a quasiparticle in the normal-metal region

$$\int_{-d/2}^{d/2} dx \sum_n [|w_n^+(kt)|^2 + |w_n^-(kt)|^2] = P_N(E_k)/S , \quad (15)$$

where S is the area of the junction. The probability $P_N(E_k)$ For $|E_k|<\Delta$ here has the form $(1+f)^{-1}$, where

$$f = 2\frac{\int_{-d/2}^{L/2} dz |\psi_s|^2}{\int_{-d/2}^{d/2} dz |\psi_n|^2} = \frac{v_n \xi}{\Delta d\sqrt{1-\varepsilon^2}} r[1+(r^2-1)|\Omega(\varepsilon)|^2]^{-1} \quad (16)$$

and L is the thickness of the SNS junction. In the limit $r \to 1$, f coincides with the result of Ref. 9. When $|E_k|>\Delta$, this probability is equal to the ratio of momentum-dependent equilibrium densities of states of the normal metal and superconductor $P_N(E_k) = (d/L)f_1$, where f_1 can be found using the stationary Green function[14,15] solution of Eq. (1). For energies near the gap and for a width of the weak region d at which a new quasidiscrete level appears (the last condition follows from the equation $\cos\phi - \cos 2d_0 = 0$), we get

$$f_1 = \rho/r \quad (17)$$

and $f_1 = 2r\rho/(1+r^2)$ when $|E_k| \gg \Delta$. If $v_n = v_s$ and $m_n = m_s$ then $f_1 = 1$.

To evaluate the quadratic terms from the current (14) and normalization condition (15), as was said before, we neglect the overlap between different n,

$$|w^{\pm}(kt)|^2 = \sum_n |w_n^{\pm}(kt)|^2 = \sum_n \left| \int_{2un_k}^{2u(1+n_k)} d\varepsilon\, C^{\pm}(\varepsilon+un) w_N^{\mp}(\varepsilon+un) \right|^2 , \quad (18)$$

where the integration is done over a chosen energy interval (k) of width 2 eV. Due to the Gaussian form of $C_1^{\mp}(\varepsilon)$, we change E (or ε) to $E_k \pm eV/2$ (or $\varepsilon_\kappa \pm u/2$, $\varepsilon_k = E_k/\Delta$) in the preexponent coefficient of the matrix product (9). The sign depends by which factor $C_1^+(E)$ or $C_1^-(E)$ these products are multiplied. Thus the integration in Eq. (18) will touch only the exponents. We can write the matrix product from Eq. (9) as follows:

$$\begin{bmatrix} C^+(\varepsilon+um) \\ C^-(\varepsilon+um) \end{bmatrix} = \sum_{p=1}^{m} y^{2p-m} \begin{bmatrix} X_+^a[p,m] + Z_-^a[p,m]y^{-1} \\ Z_+^a[p,m]y^{-1} + X_-^a[p,m]y^{-2} \end{bmatrix} , \quad (19)$$

$$\begin{bmatrix} C^+(\varepsilon-um) \\ C^-(\varepsilon-um) \end{bmatrix} = \sum_{p=1}^{m} y^{2p-m} \begin{bmatrix} X_+^b[p,m]y^{-2} + Z_-^b[p,m]y^{-1} \\ Z_+^b[p,m]y^{-1} + X_-^b[p,m] \end{bmatrix} ,$$

where $y = \exp(d_0\varepsilon)$, the indices a,b relate X,Z to the matrix $M_{a,b}$, and the signs $+,-$ of X,Z show that the energy variable is $\varepsilon_k + u$ or $\varepsilon_k - u$, respectively. $X_+^a[p,m]$ and $Z_-^a[p,m]$ obey the recursion relations [which can be directly obtained from Eq. (9)]

$$\begin{aligned} X_+^a[p,m] &= \gamma^-(\varepsilon_k+um) X_+^a[p-1,m-1] \\ &\quad + i\sqrt{r^2-1}\,\Omega(\varepsilon_k+um) \\ &\quad \times \exp(iS_m^+) Z_+^a[p,m-1] , \\ Z_+^a[p,m] &= -i\sqrt{r^2-1}\,\Omega(\varepsilon_k+um) \\ &\quad \times \exp(-iS_m^+) X_+^a[p-1,m-1] \\ &\quad + \gamma^+(\varepsilon_k+um) Z_+^a[p,m-1] . \end{aligned} \quad (20)$$

The analogous equations describe the remaining quantities $X_-^a[p,m]$, $Z_-^a[p,m]$, $X_\pm^b[p,m]$, and $Z_\pm^b[p,m]$ (see the appendix). The Andreev current [formula (14)] and the normalization condition [Eq. (15)] can be expressed in terms of these new quantities. We begin with Eq. (15); substituting the coefficients C^+,C^- which are given by Eq. (19) into sum (18) and (15), we get

$$\sum_{m=0}^{n} \langle y^m \rangle^2 [W^+(m,n) + W^-(m,n) + 2\delta_{m,0}] N^2(t) = P_N(E_k)/S . \quad (21)$$

Here

$$\begin{aligned} W^+(m,n) = \sum_{j=m}^{n} \{ & |X_+^a[(m+j)/2,j]|^2 \\ & + |Z_-^a[(m+j)/2+1,j+1]|^2 \\ & + |X_+^b[(m+j)/2+2,j+2]|^2 \\ & + |Z_-^b[(m+j)/2+1,j+1]^2\} , \end{aligned}$$

$$\begin{aligned} W^-(m,n) = \sum_{j=m}^{n} \{ & |X_-^b[(m+j)/2,j]|^2 \\ & + |Z_+^b[(m+j)/2+1,j+1]|^2 \\ & + |Z_+^a[(m+j)/2+1,j+1]|^2 \\ & + |X_-^a[(m+j)/2+2,j+2]|^2\} , \end{aligned}$$

where the δ symbol originates from the $n=0$ terms in the normalization condition [Eq. (15)].

$$\langle\cdots\rangle=\int_{-d/2}^{d/2}dz\int d\varepsilon\exp[i(z_0-t)\varepsilon+id_0\varepsilon/2-\varepsilon^2/\delta^2]\cdots$$

denotes the energy and space integration with Gaussian distribution [see Eq. (10)] and with the time- and z-dependent exponent factors of the $w_N^{\pm}$ amplitudes [Eq. (18)]. The summation over the j index is such that $(m+j)/2$ remains an integer and the second argument of the function X,Z does not exceed n. Here we extracted from the introduced new parameters X,Z the common energy-dependent factors [see $C_1^{\pm}(E)$] and also the time-dependent normalization constant $N(t)$. The latter can easily be found because in the approximation which we use $(ud_0>1)$, the overlap of the wave-packet solution is neglected. Therefore, for a given time only one m is important in the sum (21):

$$N^2(t)=N_m^2=[W^+(m,n)+W^-(m,n)+2\delta_{m,0}]^{-1}P_N(E_k)/S\ . \tag{22}$$

The calculation of the time-averaged Andreev part of the total current is similar to getting the formula (21). The only new moment is the procedure of averaging on time which involves the time integrations of powers of $\langle y^m\rangle$ (only the time-dependent factors) in the sum like Eq. (21). We did this in the same way as in Ref. 13. Thus the current has form [see Eq. (14)]

$$J_{\rm AR}=-\frac{e}{m_n}\frac{1}{Sd}\sum_k P_N(E_k)\{f_0(E_k)k_e-[1-f_0(E_k)]k_h\}F(E_k)\ , \tag{23}$$

$$F[E_k]=\sum_{m=0}^{n}e^{-md/l}\frac{W^+(m,n)-W^-(m,n)}{W^+(m,n)+W^-(m,n)+2\delta_{m,0}}\ ,$$

Formally the maximal number of Andreev reflections n is defined by the inelastic scattering length l, but in fact this number can be much smaller and the sum over m is limited by the field voltage eV. Nevertheless, if the SNS weak links are formed by thin microshorts in the tunnel junctions then additional inelastic channels are opened, decreasing l. Therefore l of the order of d is possible for $d\le\xi_0$. This can be the case which was considered in Ref. 19 where the inelastic length was of the order of the hopping length. (For InO this length is 210 Å, for a-Ge, 100 Å.) The exponential decrease of the $2\Delta/n$ peak amplitude with the Andreev reflection order n was shown in Ref. 19 to be effective for small n.

In formulas (21)–(23) the fast oscillations in atomic distances remain because of the phase factors $\exp(iS_j^{\pm})$. It is possible to separate these factors in the general form of the X,Z values and to take an average on the atomic scale, but it is simpler for a small number of Andreev reflections n to calculate w^+ and w^- directly; after that we can get rid of the fast atomic-scale oscillations. A closed analytic form for $W^{\pm}(m,n)$ can be obtained for a small normal reflection rate, i.e., in the first order on r^2-1. This normal or elastic reflection is defined by mismatches between the Fermi velocities and effective masses of superconductors from the banks of the SNS junction and the normal-metal weak link. For arbitrary normal reflection we get from Eq. (20)

$$\begin{aligned}
&X_+^a[1,1]=\gamma^-(\varepsilon_k+u)\ ,\\
&Z_+^a[1,1]=i\sqrt{r^2-1}\,\Omega(\varepsilon_k+u)\exp(-iS_1^+)\ ,\\
&X_+^a[m,m]=\prod_{j=1}^{m}\gamma^-(\varepsilon_k+uj)\ ,\\
&Z_+^a[m,m]=i\sqrt{r^2-1}\,\Omega(\varepsilon_k+mu)\exp(-iS_m^+)\times X_+^a[m-1,m-1],\quad m>1\ ,\\
&Z_+^a[1,m]=Z_+^a[1,1]\prod_{j=2}^{m}\gamma^+(\varepsilon_k+uj)\ ,\\
&X_+^a[1,m]=-i\sqrt{r^2-1}\,\Omega(\varepsilon_k+mu)\exp(iS_m^+)\times Z_+^a[1,m-1]\ ,\quad m>1\ .
\end{aligned} \tag{24}$$

Similar expressions follow for the remaining variables X^a, Z^a, $Z_{\pm}^b$, and $X_{\pm}^b$ (see the Appendix). Now we take the limit of small normal reflection $|r^2-1|\ll1$ in which all quantities can be calculated exactly. We have in the $(a,+)$ case

$$|X_+^a[m,n]|^2=\delta_{m,n}|X_+^a[m,m]|^2\ ,$$

$$Z_+^a[1,m]=(r^2-1)\prod_{j=m+1}^{n}|\gamma^+(\varepsilon_k+uj)|^2|\Omega(\varepsilon_k+mu)|^2\times\prod_{j=1}^{m-1}|\gamma^-(\varepsilon_k+uj)|^2\ . \tag{25}$$

Thus finally the functions which represent the dynamics of quasiparticles in the normal metal, $W^{\pm}(m,n)$, for weak elastic reflection are reduced to the summation of the product of Andreev reflection probabilities and the one specular reflection

$$\begin{aligned}
W^+(m,n)&=W^-(m,n)(u\to-u)\\
&=\prod_{j=1}^{m}|\gamma^-(\varepsilon_k+uj)|^2+(r^2-1)\sum_{i=m}^{n-1}\left\{\prod_{j=1}^{(i+m)/2}|\gamma^-[\varepsilon_k+u(i-j+1)]|^2|\Omega[\varepsilon_k+u(i-m)/2]|^2\right.\\
&\quad\times\prod_{j=1}^{(i-m)/2}|\gamma^+[\varepsilon_k+u(j-1)]|^2\\
&\quad+\prod_{j=2+(i+m)/2}^{i+1}|\gamma^+(\varepsilon_k-uj)|^2|\Omega\{\varepsilon_k-u[(i+m)/2+1]\}|^2\\
&\quad\left.\times\prod_{j=1}^{(i+m)/2}|\gamma^-(\varepsilon_k-uj)|^2\right\}\ ,
\end{aligned} \tag{26}$$

where the summation is provided over those i which satisfy the condition that $(i+m)/2$ is an integer. When the upper indices of the products in the sums are less than the lower ones by 1, the corresponding functions are substituted by 1. These expressions define the Andreev current for the weak elastic scattering of quasiparticles at SN interfaces. In spite of their complicated form, they have a simple physical meaning: to the flow of quasiparticles moving in one direction, those represented by Andreev amplitude γ^-, a contribution also comes from quasiparticles moving in the opposite direction after they are elastically reflected from the SN boundary (amplitude γ^+).

To complete the calculation of the Andreev current we need to do a summation over the energy spectrum. For this we transform the sum over k into an integral over density of states. If $E_k > \Delta$ then this density is equal to the usual BCS density of states. When $|E_k| < \Delta$, the spectrum is multiband and quasidiscrete [depends on the momentum components parallel to the interfaces (here from ξ)]. The density of states depends on the subband index i

$$g_i(E)=\frac{S}{2\pi^2}\int d^2k\,\delta(E-E_{ik})\ . \tag{27}$$

The dispersion equation [Eq. (8)] is much more complicated than in the case $v_n=v_s$, $m_n=m_s$ for which the simple analytical formula for $g_i(E)$ was obtained (see Ref. 20). Here the solution can be written as the dependence ξ on the quasiparticle energy E_{ik}. The total current density is the sum of the Andreev and the usual Ohmic current density

$$J=J_{\mathrm{AR}}+J_n\ .$$

J_{AR} is the integral over the energy:

$$J_{\mathrm{AR}}=\tfrac{1}{2}ev_nN_n\Delta\left[\pi\alpha\sum_i\int_0^{\varepsilon_{i\max}}d\varepsilon\,\xi_i^2(\varepsilon)\tanh(\varepsilon/2T)\left|\frac{\partial\varepsilon_k}{\partial\xi_i}\right|^{-1}_{\varepsilon_k=\varepsilon}P_N(\varepsilon)F(\varepsilon)\right.$$
$$\left.+\frac{N_S}{N_n}\int_1^\infty d\varepsilon\tanh(\varepsilon/2T)\frac{\varepsilon}{\sqrt{\varepsilon^2-1}}\int_0^1 d\eta\,\xi f_1F(\varepsilon)\right]\ . \tag{28}$$

Here ξ and η were given earlier [see text before Eq. (2)], $\alpha=\hbar v_n/(d\Delta)$, and $P_N(\varepsilon)$ and f_1 are expressed by the formulas (16) and (17). We approximated as well the term in square brackets from Eq. (23) by $-m_nv_n\xi\tanh(\varepsilon/2T)$ and introduced the densities of states at the Fermi energy of the superconductor N_s and weak link N_n. The Ohmic current density is

$$J_n=\frac{V}{RS}\ , \tag{29}$$

where R is the normal-state resistance of the junction. Taking f_1 from Eq. (17) with the parameter r as a constant and $m_sv_s<m_nv_n$, we estimate this resistance to be

$$\frac{1}{RS}=\frac{1}{r}(v_se^2N_s)v\gamma^2\alpha\ . \tag{30}$$

Here α/v_n is the average over the superconductor Fermi surface of the inverse z component of velocity,

$$\alpha=\frac{\sum_k f_0(E_k)\eta/\xi^2}{3\sum_k f_0(E_k)}\ . \tag{31}$$

The relation between η and ξ was given before [see text after Eq. (1)].

The density of bound Andreev states crucially depends upon the parameter α and so on the thickness d. For a large thickness of the weak link, this density of states is small compared with the BCS density of states. Therefore the discrete spectrum weakly influences the Andreev current. Only the contribution from the continuum spectrum $|E|>\Delta$ is important. In the opposite limit $d<\xi_0$, the density of bound Andreev states is increased; however, in this case for the voltages $eV<\Delta$ the oscillations with frequencies $\sim 2ud_0$ become relevant and our theory is not applicable. The other point we note is the lifting of the degeneracy of levels which takes place in the case $r=1$ due to the degeneracy of states with opposite z momenta of equal magnitude. When r deviates from 1, the twice degenerate quasidiscrete energy bands are split into two branches.

IV. CONCLUSION

In conclusion, we calculated the current-voltage characteristics of superconductor-normal-metal-superconductor junctions due to the multiple Andreev and normal elastic reflections at the SN interfaces. The normal reflections of quasiparticles appear here due to the mismatches in Fermi velocities and effective masses of superconductors and weak link. We used the wave-packet solution of the nonequilibrium time-dependent Bogoliubov–de Gennes equations and unlike Refs. 9 and 10 applied the boundary conditions on the wave function of the quasiparticle. At the voltages of interest for multiple AR ($eV<2\Delta$), our results are valid for rather thick weak links. At these voltages the I-V characteristic reveals a subharmonic energy-gap structure which has been observed in superconductor-semiconductor-superconductor (S-Sm-S) systems.[2,21] In Ref. 21 pure material S-Sm-S junctions were prepared. A very sharp dip near zero voltage in dV/dJ was observed. We think that such enhancement at $V\to 0$ (when our theory is not applicable) is typical for weak links with finite conductivity for junctions with a supercurrent,[4,5] though direct calculations are required. For large voltage $eV \gg \Delta$ we have excess current. It is defined by one Andreev reflection, and the main contribution comes from the second integral in Eq. (28). We have

$$J_{ex} = \frac{ev_n\Delta}{2} N_s \int_1^\infty d\varepsilon \int_0^1 d\eta \tanh(\varepsilon/2T) \frac{\varepsilon}{\sqrt{\varepsilon^2-1}} \times \xi f_1 F_{ex}(\varepsilon) ,$$

where using Eq. (24) we find, considering the previously defined rule for energies exceeding the gap [see the text after Eq. (10)],

$$F_{ex}(\varepsilon) = \frac{|\gamma^-(\varepsilon+u)|^2 - |\gamma^-(\varepsilon-u)|^2}{|\gamma^-(\varepsilon+u)|^2 + |\gamma^-(\varepsilon-u)|^2} + \frac{(|\Omega(\varepsilon-u)|^2 - |\Omega(\varepsilon)|^2)(r^2-1)}{2+[|\Omega(\varepsilon-u)|^2+|\Omega(\varepsilon)|^2](r^2-1)}$$
$$\approx -1 + \frac{|\Omega(\varepsilon-u)|^2(r^2-1)}{2+|\Omega(\varepsilon-u)|^2(r^2-1)} .$$

From this equation and Eq. (24) the excess current multiplied by r decreases as a function of the reflection parameter r.

Formula (23) for the Andreev current was obtained for the pure SNS junction. It involves only the inelastic scattering length and therefore we expect that this formula qualitatively describes weak links which are nearly at the metal-insulator transition with inelastic length of the order of the hopping length. In this case the exponential decrease of the $2\Delta/n$ peak amplitude with the Andreev reflection order n is effective for small number n, as was observed in Ref. 19.

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APPENDIX

Here are represented the complete set of recursive relations for functions $X^a_-[p,m]$, $Z^a_-[p,m]$, and $X^b_\pm[p,m]$. From Eq. (9) we get

$$X^b_+[p,m] = \gamma^+[\varepsilon_k - u(m-1)]X^b_+[p,m-1] - i\sqrt{r^2-1}\,\Omega[\varepsilon_k - u(m-1)] \times \exp(-iS^-_m)Z^b_+[p-1,m-1] ,$$
$$Z^b_+[p,m] = i\sqrt{r^2-1}\,\Omega[\varepsilon_k - u(m-1)] \times \exp(iS^-_m)X^b_+[p,m-1] + \gamma^-[\varepsilon_k - u(m-1)]Z^b_+[p-1,m-1] , \quad \text{(A1)}$$

$$X^b_-[p,m] = \gamma^-(\varepsilon_k - um)X^b_-[p-1,m-1] + i\sqrt{r^2-1}\,\Omega(\varepsilon_k - um) \times \exp(iS^-_m)Z^b_-[p,m-1] ,$$
$$Z^b_-[p,m] = -i\sqrt{r^2-1}\,\Omega(\varepsilon_k - um) \times \exp(-iS^-_m)X^b_-[p-1,m-1] + \gamma^+(\varepsilon_k - um)Z^b_-[p,m-1] , \quad \text{(A2)}$$

$$X^a_-[p,m] = \gamma^+[\varepsilon_k + u(m-1)]X^a_-[p,m-1] - i\sqrt{r^2-1}\,\Omega[\varepsilon_k + u(m-1)] \times \exp(-iS^+_m)Z^a_-[p-1,m-1] ,$$
$$Z^a_-[p,m] = i\sqrt{r^2-1}\,\Omega[\varepsilon_k + u(m-1)] \times \exp(iS^+_m)X^a_-[p,m-1] + \gamma^-[\varepsilon_k + u(m-1)]Z^a_+[p-1,m-1] . \quad \text{(A3)}$$

Initial values of these functions are

$$X^a_-[1,1] = \gamma^-(\varepsilon_k) , \quad X^b_+[1,1] = \gamma^+(\varepsilon_k) ,$$
$$X^b_-[1,1] = \gamma^-(\varepsilon_k - u) ,$$
$$Z^a_-[1,1] = i\sqrt{r^2-1}\,\Omega(\varepsilon_k)\exp(iS^+_1) ,$$
$$Z^b_+[1,1] = i\sqrt{r^2-1}\,\Omega(\varepsilon_k)\exp(iS^-_1) ,$$
$$Z^b_-[1,1] = -i\sqrt{r^2-1}\,\Omega(\varepsilon_k - u)\exp(-iS^-_1) .$$

Other relations similar to those of Eq. (24) can easily be obtained.

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