Optimization of superconductor–normal-metal–superconductor Josephson junctions for high critical-current density

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The application of superconducting $Bi_2Sr_2CaCu_2O_8$ and $YBa_2Cu_3O_7$ wires or tapes to electronic devices requires the optimization of the transport properties in Ohmic contacts between the superconductor and the normal metal in the circuit. This paper presents results of tunneling theory in superconductor–normal-metal–superconductor (SNS) junctions, in both pure and dirty limits. We derive expressions for the critical-current density as a function of the normal-metal resistivity in the dirty limit or of the ratio of Fermi velocities and effective masses in the clean limit. In the latter case the critical current increases when the ratio γ of the Fermi velocity in the superconductor to that of the weak link becomes much less than 1 and it also has a local maximum if γ is close to 1. This local maximum is more pronounced if the ratio of effective masses is large. For temperatures well below the critical temperature of the superconductors the model with abrupt pair potential on the SN interfaces is considered and its applicability near the critical temperature is examined.

I. INTRODUCTION

For many technical applications of superconductivity, a large critical current is an essential property. The integration of superconducting devices in electronic circuits requires that the current-carrying ability be preserved in electrode Ohmic contacts with metals, or at metallic-superconducting interfaces within the superconducting device. Such contacts are present when the necessity for improved ductility of the superconducting ceramic material leads to the use of metal-matrix composites such as $Bi_2Sr_2CaCu_2O_8$ (BSCCO) granules dispersed in an Ag matrix, or BSCCO powder in an Ag tube.[1] Composite wires or tapes can be modeled as a dispersion of strongly superconductive granular islands embedded in a metallic matrix such that the superconducting order parameters are coupled by Josephson junctions. Each such junction is then a superconductor–normal-metal–superconductor (SNS) junction; the normal-metal part is also referred to as a weak link. The critical current of such SNS junctions determines the critical current of the composite system.

It is therefore of considerable interest to identify which material parameters affect the SNS critical current and eventually optimize the Josephson junctions by the properties (physical, chemical or geometrical) of the normal metal that could be adjusted to maximize the critical-current density.

A direct Josephson effect in high-T_c SNS junctions has been observed in many experiments (see review article[2] for a list of references). The theoretical study of SNS-type Josephson junctions considers different temperature intervals below T_c and various purity limits for materials which compose the SNS trilayer. For T close to T_c, SNS junctions were studied in the dirty limit,[2–4] i.e., in the case when the mean free paths of quasiparticles in the superconductor l_s and in the weak link l_n are less than the corresponding coherence lengths. A pure SNS sandwich with abrupt pair potential barriers at SN interfaces (valid for $T \ll T_c$) has been considered by Ishii[5] at $T=0$ and by Bardeen and Johnson.[6] In this model the effective mass m_s, m_n and Fermi velocities v_s, v_n for the superconductors and for the weak link, respectively, are the same. Near T_c self-consistent solutions of the order parameter show that the pair potential is changed in a large region near the SN boundary, i.e., the so-called proximity effect. This has a significant impact on the Josephson current affecting also its temperature dependence. The model with abrupt pair potential barriers at SN interfaces needs then to be modified. The proximity effect of the pure SNS junction is known[7] for $v_s = v_n$ and $m_s = m_n$ and to some extent for more general cases,[8,9] The proximity effect with $v_s \neq v_n$ but with $m_s = m_n$ was considered by Kieselmann[10] who obtained numerically the pair potential for several values of temperature. We have used Gor'kov equations to calculate the critical Josephson current, though the approach based on the quasiclassical theory of superconductivity is possible. Quasiclassical equations and boundary conditions general enough to describe a wide class of superconductors with magnetic active interfaces were obtained recently by Millis, Rainer, and Sauls.[11] This theory was generalized on superconductor–normal-metal double-layer system in the article.[12]

In this work we evaluate the critical Josephson current of both pure and dirty SNS junctions in the present case where $v_s \neq v_n$ and $m_s \neq m_n$. The only assumption made is that the thickness d of the weak link is large compared with its coherence length, i.e., weak tunneling. In Sec. II the pure SNS junction at temperatures near T_c is studied by solving the integral equation for the order parameter and the critical current; the weak link is taken as a normal metal with a critical temperature $T_{cn}=0$. Optimum values of the parameters v_s/v_n and m_s/m_n for maximiz-

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ing the critical current are found. In Sec. III dirty SNS junctions at temperatures near T_c are studied; numerical data by Barone and Ovchinnikov[3] are analyzed and simple analytic expressions for the critical current and pair potential is given. This allows for an easy parameter optimization (e.g., resistivity of the weak link) for maximizing the critical current. Section IV solves the pure SNS system in the case when the critical temperature of the weak link deviates from zero and comparison with results for the dirty limit are made. In Sec. V pure SNS junctions at temperatures well below T_c are studied by using the model with abrupt pair potential on an SN interface,[5,6] generalized to $v_s \neq v_n$ and $m_s \neq m_n$. It is shown that under some conditions (large difference in masses and velocities) the results are applicable even if T is close to T_c. In the conclusions, Sec. VI, we compare our pure limit results with experimental data[13] showing unusual temperature dependence of the critical current. Appendix A solves for the proximity effect, while Appendix B discusses an instructive case of reflection from a single SN boundary.

II. PURE SNS JUNCTION

SNS devices working in the pure limit have been fabricated with superconductive $YBa_2Cu_3O_7$ (YBCO) thin-film electrodes and Ag normal-metal bridges.[13] The clean limit has also been achieved for Bi-based high-T_c superconductors. In this case $l_s > \xi_s = v_s/2\pi T_c$, $l_n > \xi_n = v_n/2\pi T$, where ξ_s and ξ_n are coherence lengths for the superconductor and normal metal, respectively and T_c is the critical temperature of the superconductors. We consider the case that the temperature T is close to T_c and that the critical temperature in the weak limit is $T_{cn}=0$ (Sec. IV considers $T_{cn}\neq 0$). The relevant physical parameters are m_s, m_n, v_s, and v_n. Superconductors and normal metal in the junction have the same chemical potential (see Fig. 1). For such trilayers SNS structure the total Hamiltonian has the form

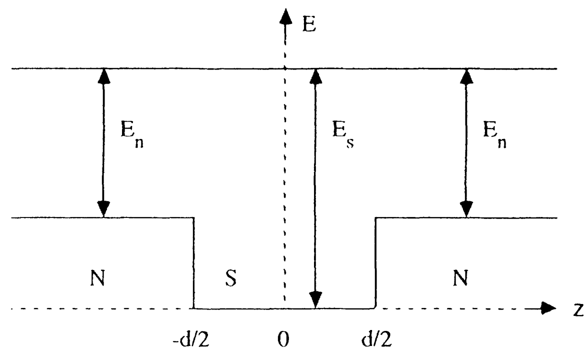

FIG. 1. Schematic energy diagram for SNS junction in nonsuperconducting state. $E_{s,n} = m_{s,n} v^2_{s,n}/2$ are the Fermi energies of the superconductors and weak link, respectively.

$$H_{sns} = \int \frac{d^2p}{(2\pi)^2} \int_{-\infty}^{\infty} dz \left[\left(\frac{p^2}{2m} - \frac{mv^2}{2} \right) \psi^\dagger_{\alpha p} \psi_{\alpha p} + \frac{1}{2m} \frac{d\psi^\dagger_{\alpha p}}{dz} \frac{d\psi_{\alpha p}}{dz} + \tilde{\Delta}(z) \psi^\dagger_{\uparrow p} \psi^\dagger_{\downarrow -p} + \tilde{\Delta}^*(z) \psi_{\downarrow p} \psi_{\uparrow -p} \right] .$$

The parameters in H_{sns} are z dependent: m, v are equal to m_s, v_s if $|z| > d/2$, or to m_n, v_n if $|z| < d/2$; the relevant energies are shown in Fig. 1. In H_{sns}, $\psi_{\alpha p}$, $\psi^\dagger_{\alpha p}$ are the quasiparticle second quantization operators, $\alpha = \uparrow, \downarrow$ is the spin index, and the sum on α is implied. $\tilde{\Delta}(z)$ stands for the z-dependent order parameter. A term $\sim |\tilde{\Delta}|^2$, which has no dependence on quasiclassical operators, is omitted in the expression for the Hamiltonian H_{sns}. We have used the homogeneity of the junction in the x,y directions (parallel to the junction) and made Fourier transformation of the ψ operators on these coordinates. The corresponding momentum component p of the quasiparticle is continuous at the SN interface due to this homogeneity.

Consider first the Green's function of the SNS when the all components of the junction are in the nonsuperconducting state, i.e., $\tilde{\Delta}(z)=0$ in H_{sns}. The Green's function which depends only on the z coordinate obeys then the equation:

$$\left[i\omega + \frac{1}{2} \frac{d}{dz} \left(\frac{1}{m} \frac{d}{dz} \right) - \frac{p^2}{2m} + \frac{mv^2}{2} \right] G_{\omega p}(z,z') = \delta(z-z') , \quad (1)$$

with the boundary conditions that the values:

$$G_{\omega p}(z,z'); \quad (1/m)(d/dz) G_{\omega p}(z,z') ;$$

are continuous on z at each interface of SNS (see Fig. 1). Due to the δ function in the right-hand side of Eq. (1) there is also the relation

$$\frac{d}{dz} G_{\omega p}(z,z')_{z\to z'+0} - \frac{d}{dz} G_{\omega p}(z,z')_{z \to z'-0} = 2m . \quad (2)$$

In formulas (1)–(3) $\omega = \pi T(2n+1)$ with $n=0, \pm 1, \pm 2, \ldots$ are discrete fermion frequencies.

The Green's function can be written in the form

$$G_{\omega p}(z,z') = C[\theta(z-z') w(z) u(z') + \theta(z'-z) w(z') u(z)] . \quad (3)$$

Here $\theta(x)$ is the Heaviside step function which is equal to 1, if $x>0$ or to 0 when $x<0$. The functions $w(z)$, $u(z)$, satisfy the homogeneous Eq. (1). They, as well as $(1/m)w'(z)$ and $(1/m)u'(z)$ (prime stands for a derivative on z) are continuous at the SN surfaces. Furthermore, for $z \to \infty$ $u(-z) \to 0$, $w(z) \to 0$. The constant C is obtained from Eqs. (2) and (3),

$$C = 2m \left[\frac{dw(z')}{dz'} u(z') - w(z') \frac{du(z')}{dz'} \right]^{-1} ,$$

The functions $w(z)$, $u(z)$ for three different regions of z

can be easily found:

$$u(z)=\exp[k_s z],\quad z<-d/2$$
$$=\frac{[\exp(k_n(z+d/2))-\alpha(\xi)\exp(-k_n(z+d/2))]\exp(-k_s d/2)}{[1-\alpha(\xi)]},\quad |z|<\frac{d}{2}$$
$$=\frac{\exp(k_s(z-d))[\exp(k_n d)-\alpha^2(\xi)\exp(-k_n d)]+2\alpha(\xi)\exp(-k_s z)sh(k_n d)}{[1-\alpha^2(\xi)]},\quad z>\frac{d}{2} \tag{4a}$$

$$w(z)=\exp(-k_s z),\quad z>d/2$$
$$=\frac{[\exp(-k_n(z-d/2))-\alpha(\xi)\exp(k_n(z-d/2))]\exp(-k_s d/2)}{[1-\alpha(\xi)]},\quad |z|<\frac{d}{2}$$
$$=\frac{\exp(-k_s(z+d))[\exp(k_n d)-\alpha^2(\xi)\exp(-k_n d)]+2\alpha(\xi)\exp(k_s z)sh(k_n d)}{[1-\alpha^2(\xi)]},\quad z<-\frac{d}{2} \tag{4b}$$

where instead of p we introduced a new variable $\xi=1-p^2/(m_n v_n)^2$, and the notations $k_n^2=p^2-(m_n v_n)^2-2im_n\omega$; $k_s^2=p^2-(m_s v_s)^2-2im_s\omega$; the real parts of k_s and k_n are taken to be positive. The new function $\alpha(\xi)$ that appears in the equations for w and u describes the reflection of quasiparticle from the SN boundary:

$$\alpha(\xi)=\left[\frac{k_s}{m_s}-\frac{k_n}{m_n}\right]\Bigg/\left[\frac{k_s}{m_s}+\frac{k_n}{m_n}\right]. \tag{5}$$

The single SN boundary is considered in Appendix B, where it is also shown that the reflection is minimal approximately at $\gamma=1$ where $\gamma=v_s/v_n$; more generally $\alpha(\xi)=0$ when both $\gamma=\nu=1$ where $\nu=m_s/m_n$.

The Green's function in the various intervals of z, z' can be obtained from Eqs. (3) and (4). Aiming at the calculation of the critical current in the case $d>\xi_n$, we write down the Green's function for the whole SNS system only for $z>d/2$, $z'<-d/2$. This corresponds to the approximation when the mutual influence of superconductive electrodes in the SNS junction is small; this is due to the exponentially small factor $\exp(-k_n d)$ when $d>\xi_n$. Thus we have for the Green's function of the SNS system:

$$G_{\omega p}(z,z')=-\frac{m_n}{k_n}(1-\alpha(\xi))\times\exp[k_s(z'-z+d)-k_n d],\quad z>d/2,\ z'<-d/2. \tag{6}$$

The expressions for Green's functions in other regions of z,z' have a form:

$$G_{\omega p}(z,z')=-\frac{m_n}{k_n}(1-\alpha(\xi))\times\exp\left[-k_s\left[z-\frac{d}{2}\right]-k_n\left[\frac{d}{2}-z'\right]\right]\times\{1-\alpha(\xi)\exp[-k_n(d+2z')]\},\quad z>\frac{d}{2},\ |z'|<\frac{d}{2}; \tag{7}$$

$$G_{\omega p}(z,z')=-\frac{m_n}{k_n}(1-\alpha(\xi))\times\exp\left[k_s\left[\frac{d}{2}+z'\right]-k_n\left[\frac{d}{2}+z\right]\right]\times\{1-\alpha(\xi)\exp[-k_n(d-2z)]\},\quad |z|<\frac{d}{2},\ z'<-\frac{d}{2}; \tag{8}$$

In Eqs. (7) and (8) the term proportional to $\exp(-k_n d)$ in the last brackets may be ignored when the critical current is calculated.

The Green's function Eq. (6)–(8) has contributions of two types: The first corresponds to real terms in the exponent, while the second to imaginary ones. The latter result in oscillations of $G_{\omega p}$ on atomic $\sim(1/mv)$ distances. We can simplify $\alpha(\xi)$ by neglecting in Eq. (5) terms of the order $(m_s v_s \xi_s)^{-1}$, $(m_n v_n \xi_n)^{-1}$,

$$\alpha(\xi)=\frac{[\gamma\sqrt{\eta/\xi}-1]}{[\gamma\sqrt{\eta/\xi}+1]},\quad \eta=1-\frac{1-\xi}{\nu^2\gamma^2},\quad \nu=\frac{m_s}{m_n},\quad \gamma=\frac{v_s}{v_n}. \tag{9}$$

Density of the critical current is related with the Green's function as follows:[14]

$$J=\frac{e}{im}(\nabla_r-\nabla_{r'})_{r\to r'}T\sum_\omega G^s_\omega(r,r'), \tag{10}$$

or in the case when only z dependence is essential

$$J=\frac{e}{im}(\nabla_{z'}-\nabla_z)T\sum_\omega\int\frac{d^2p}{(2\pi)^2}e^{-i\omega\delta}G^s_{\omega p}(z,z')_{z\to z'}, \tag{11}$$

where the index s stands for the full Green's function which includes also superconductive correlations, $\delta\to+0$. Near T_c, expansion to second order in $\Delta(z)$ can be applied[14,15] to $G^s_{\omega p}(z,z')$:

$$G^s_{\omega p}(z,z')=G_{\omega p}(z,z')-\int_{-\infty}^{\infty}dz_1\int_{-\infty}^{\infty}dz_2 G_{\omega p}(z,z_1)\tilde{\Delta}(z_1)G_{-\omega p}(z_2,z_1)\tilde{\Delta}(z_2)G_{\omega p}(z_2,z') , \tag{12}$$

where

$$\tilde{\Delta}(z)=\Delta(z)\left\{\theta\left[z-\frac{d}{2}\right]\exp(i\phi_1)+\theta\left[-\left[\frac{d}{2}+z\right]\right]\exp(i\phi_2)\right\}\equiv\Delta(z)f , \tag{12'}$$

Here ϕ_1,ϕ_2 are phases of the order parameter in left and right superconductors, respectively; $\Delta(z)$ is its absolute value. Inserting (12) into the formula for current density Eq. (11) we have

$$J(z=0)=j\sin\phi,\quad \phi=\phi_1-\phi_2$$

$$j=\frac{4e}{m_n}T\sum_{\omega>0}\int_{-\infty}^{-d/2}dz_1\int_{d/2}^{\infty}dz_2\int\frac{d^2p}{(2\pi)^2}\Delta(z_1)\Delta(z_2)G_{-\omega p}(z_2,z_1)\hat{L}(z_2,z_1) , \tag{13}$$

$$\hat{L}(z_2,z_1)=G_{\omega p}(0,z_1)\frac{\partial}{\partial z}G_{\omega p}(z_2,z)-G_{\omega p}(z_2,0)\frac{\partial}{\partial z}G_{\omega p}(z,z_1)_{z\to 0} ,$$

The Green's function $G_{-\omega p}(z_2,z_1)$ in the expression for j describes the motion of quasiparticle from one side of the junction to the other, So this Green's function, as was mentioned before, has to be defined for the whole SNS junction, whereas $\hat{L}(z,z')$ consists of Green's functions which describe the motion of quasiparticle from one of the superconductors to the inside of the weak link. These may be calculated for the separate SN system. We proceed further by substituting the explicit form for the Green's functions [Eqs. (6)–(9)] in formula (13) for current density j:

$$j=8em_n^2T\sum_{\omega>0}\int\frac{d^2p|1-\alpha(\xi)|^4}{4\pi^2|k_n|^2}\exp(-\tilde{k}_n d)\Delta_f^2(\tilde{k}_s) . \tag{14}$$

Here $\Delta_f(k)$ denotes the Laplace transform of $\Delta(z)$:

$$\Delta_f(\tilde{k}_s)=\int_0^\infty dz\,\Delta(z+d/2)\exp(-\tilde{k}_s z) ,$$

where

$$\tilde{k}_{n,s}=k_{n,s}+k^*_{n,s} ,$$
$$|k_n|^2=(m_n v_n)^2\xi\left[1+\frac{4\omega^2}{\xi^2v_n^2(m_nv_n)^2}\right]^{0.5} . \tag{15}$$

We can rewrite the expression for the current by integration on the variable ξ which was introduced above [see Eq. (9)]:

$$j=\frac{2}{\pi}em_n^2T\sum_{\omega>0}\int_0^1 d\xi\frac{|1-\alpha(\xi)|^4}{\xi}\exp(-\tilde{k}_n d)\Delta_f^2(\tilde{k}_s) . \tag{16}$$

The condition that the thickness of the weak link d is greater than ξ_n permits us to simplify the last expression. At T near T_c only one term in the sum is important and we can put $\xi=1$ in all terms of Eq. (16) except for the exponential term. Physically this means that the quasiparticles moving normal to the junction contribute significantly. We also ignore terms such as the last term under the square root in Eq. (15) which are proportional to $(m_nv_n\xi_n)^{-1}$. Therefore

$$\tilde{k}_n\approx\frac{2|\omega|}{v_n\sqrt{\xi}}\approx\frac{1}{\xi_n\sqrt{\xi}} ,$$
$$\tilde{k}_s=\frac{2|\omega|}{v_s\sqrt{\eta}}[2/(1+\sqrt{1+4\omega^2/v_s^4m_s^2})]^{1/2}\approx\frac{2\pi T}{v_s} . \tag{17}$$

Now we integrate (16) and find

$$j=\frac{2em_n^2v_n}{\pi^2 d}|1-\alpha|^4\Delta_f^2(\tilde{k}_s)\exp\left[-\frac{d}{\xi_n}\right]$$
$$=\frac{32em_n^2v_n}{\pi^2 d}\frac{\Delta_f^2(\tilde{k}_s)}{(1+\gamma)^4}\exp\left[-\frac{dt}{\xi_s}\gamma\right] , \tag{18}$$

where $\alpha=\alpha(\xi=1)$, $t=T/T_c$

To complete calculations of the current density, we have to obtain the Laplace transform of the order parameter at the point $\tilde{k}_s$. This means that we must know $\Delta(z)$ in the whole region near the NS boundary $0<z-d/2\lesssim 1/\tilde{k}_s$. Due to the exponent in Eq. (18), it is possible to analyze a two-layer SN system neglecting the mutual influence of superconductive banks. Thus we can define the z dependence of the order parameter considering this more simple case. Detailed derivation of $\Delta(z)$ is presented in Appendix A, while here we stress the main lines. In the vicinity of T_c, the order parameter near the SN surface on a scale of ξ_s obeys the linear integral equation of the type proposed by de Gennes for Josephson SIS junctions[16]

$$\Delta(z)=|g|2T\sum_{\omega>0}\int\frac{d^2p}{(2\pi)^2}\int_0^\infty\Delta(z')K_{\omega p}(z,z')dz' ,$$
$$K_{\omega p}(z,z')=G_{\omega p}(z,z')G_{-\omega p}(z,z'),\quad z>0,\quad z'>0 , \tag{19}$$

where g is the interaction coupling strength responsible for superconductivity. In the normal metal we assume that $g=0$ ($T_{cn}=0$), so the integration on z' in Eq. (19) is restricted to the region $z'>0$ of the superconductor. The origin $z=0$ is taken on the SN boundary. The expressions for the Green's functions and the solution of the integral equation (19) is given in Appendix A. The solution for $z\gg\xi_s$ is matched with the solution of the Ginzburg-Landau equation:

$$\Delta(z)=\Delta_0 \tanh\sqrt{\tau/2}\left[\frac{z}{\xi_{1s}}+\delta_p\right],$$
$$\xi_{1s}=\sqrt{7\zeta(3)/12}\xi_s=0.84\xi_s . \quad (20)$$

Here $\Delta_0=3.06T_c\tau^{1/2}$ is the superconducting gap in the bulk of the superconductor and $\tau=1-t$. The value of δ_p is obtained from the solution of Eq. (19); it depends on material parameters for both the superconductor and the weak link, and it determines to what extent $\Delta(z)$ is reduced by proximity to the normal metal. The solution of Eq. (19) has a form (see Appendix A)

$$\Delta(z)=\Delta_0\sqrt{\tau/2}\left[\frac{z}{\xi_{1s}}+\delta_p+a\exp\left[-\frac{z}{\xi_s}\right]\right], \quad (21)$$

where δ_p and a are given by

$$\delta_p=1.7A_2+1.52\frac{A_1^2}{B_1},\quad a=\frac{1}{D}(2.1\delta_p B_2-2.4A_2) . \quad (22)$$

This generalizes previous results.[8] All values A_i, B_i $(i=1,2)$, and D depend on the reflection coefficient $\alpha(\xi)$,

$$A_j=\frac{1}{2}\int_0^1 d\eta\eta^{0.5j}[1+|\alpha(\xi)|^2],$$
$$B_j=\frac{1}{2}\int_0^1 d\eta\eta^{0.5(j-1)}[1-|\alpha(\xi)|^2], \quad (23)$$
$$D=\frac{1}{2}\int_0^1 d\eta\sqrt{\eta}[1+|\alpha(\xi)|^2](1+\sqrt{\eta})^{-1},$$

where ξ and η are defined in Eq. (9), and B_j vanish when the reflection coefficient $\alpha(\xi)\to 1$. In the case when the reflection coefficient $\alpha(\xi)$ does not depend on ν, parameter δ_p as function of γ was calculated by another method (solving numerically the Gor'kov equations) in Ref. 9.

All integrals can be easily calculated numerically. As a result we obtain Figs. 2–4, which represent δ_p as a function of γ for various values of ν. From these figures we see that the minimum of $\delta_p(\gamma)$ depends on the effective mass ratio ν. This minimum corresponds to that of reflection from the SN surface. This behavior differs from the corresponding dependence in dirty limit (see below). This is possibly due to different characteristics reflecting how the quasiparticles move in the pure and dirty cases. In dirty limit the quasiparticles spread diffusely, while in the pure system they move ballistically. For the model[7] $\alpha(\xi)=0$, $m_n=m_s$, $v_n=v_s$ we have $\delta_p=0.764$.

Now with the help of Eq. (21) we can calculate the Laplace transform $\Delta_f(\tilde{k}_s)$ and find the amplitude of the critical-current density:

$$j=\frac{47.7\tau^2}{\gamma\nu^2 d_0}eT_cN_s t v_s F_p \exp(-d_0\gamma),\quad d_0=\frac{td}{\xi_s},$$
$$F_p=\frac{(\delta_p+1.2+0.5a)^2}{(1+\gamma)^4} . \quad (24)$$

Here the density of quasiparticle states $N_s=m_s^2 v_s/2\pi^2$ is introduced, as well as dimensionless thickness of the weak link d_0. Formula (24) is our main result for clean SNS junctions. In Figs. 5 and 6, F_p is plotted as function of γ and various values of ν. We see that in the range $0<\gamma<2$ for the most values of ν, F_p has a minimum. F_p increases when γ approaches the value 2. This is due to the balance between the reflection of quasiparticles from the SN interfaces and transmission of Cooper pairs through the SN boundaries. However, for large γ, F_p decreases if γ grows. Indeed, when $\gamma\gg 1$ the reflection coefficient $\alpha(\xi)$ is close to unity and the main contribution to δ_p comes from term with B_1 [see Eqs. (22) and (23)]. Therefore, in the limit $\gamma\gg 1$ for γ dependence of δ_p and F_p, we find $\delta_p\sim\text{const}/B_1$; $B_1\sim 2\gamma/(1+\gamma)^2 \sim 2/\gamma$; $F_p\sim\gamma^2/(1+\gamma)^4\sim 1/\gamma^2$. Such behavior of F_p shows that the suppression of the transmission probability of Cooper pairs overcomes the suppression of the proximity effect (increasing δ_p). The current j depends upon the exponential term which suppresses the current

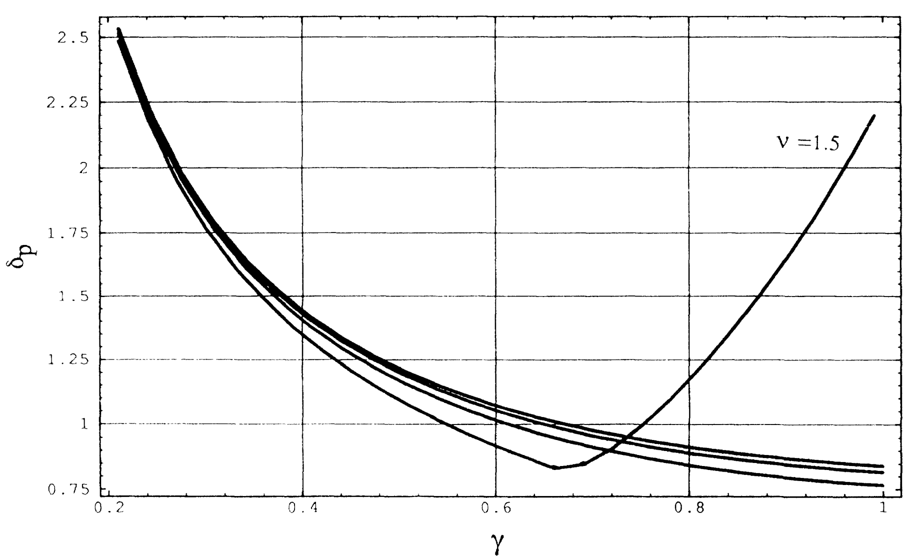

FIG. 2. The parameter δ_p [Eq. (22)] which describes the suppression of the order parameter at the SN boundary as function of γ, for different ratios ν of effective masses. The unmarked curves correspond, from top to bottom, to $\nu=0.2$, 0.6, 1.0, respectively.

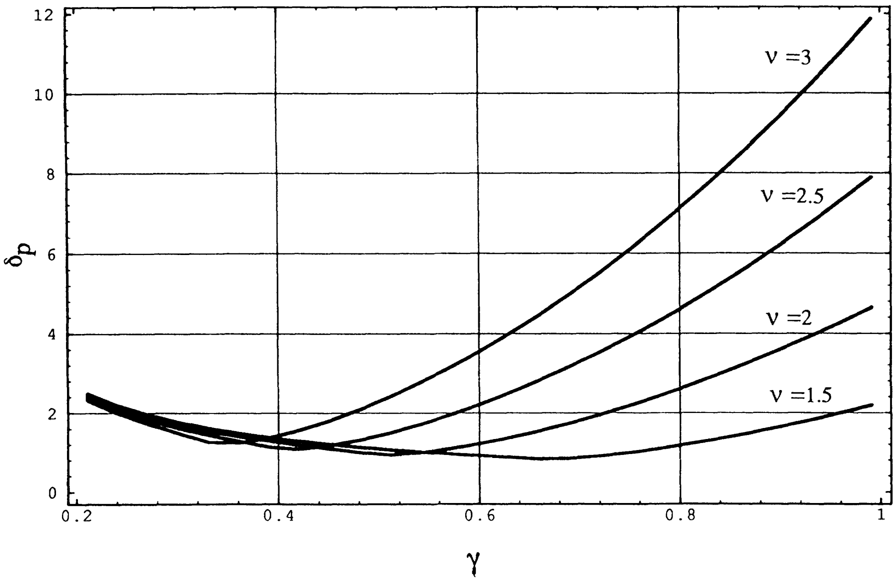

FIG. 3. The γ dependence of the parameter δ_p as in Fig. 2 for higher values of ν.

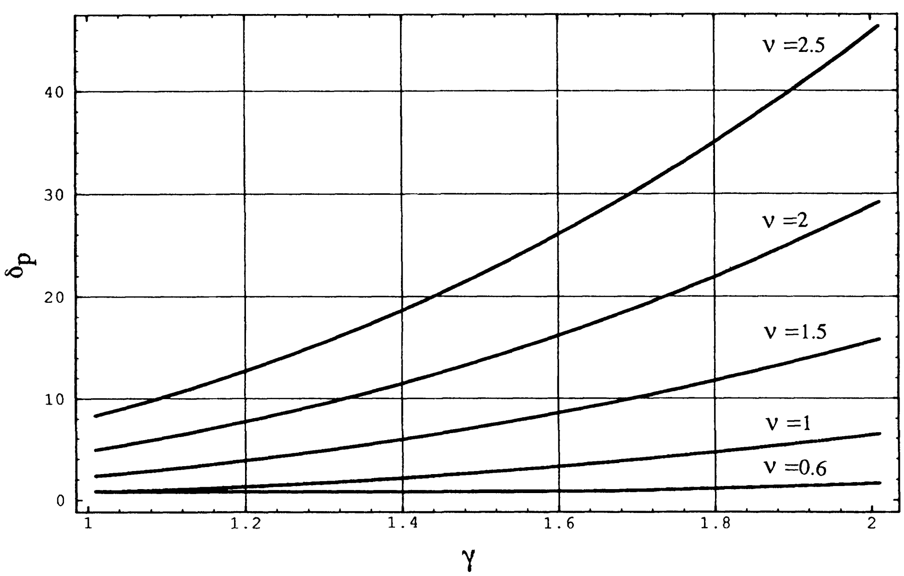

FIG. 4. δ_p as function of γ when the Fermi velocity of the superconductor exceeds that of the normal metal ($\gamma > 1$).

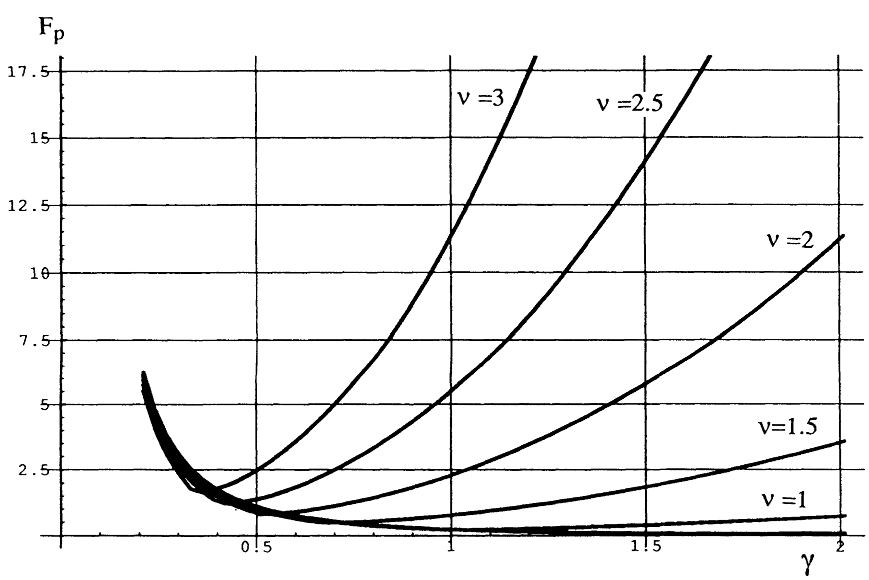

FIG. 5. The γ dependence of the pre-exponential factor F_p in the formula for the critical current density [Eq. (24)] for various ν. The lower curve corresponds to $\nu=0.2$, 0.6, 1.0. On this scale the curves representing these values of ν are close together.

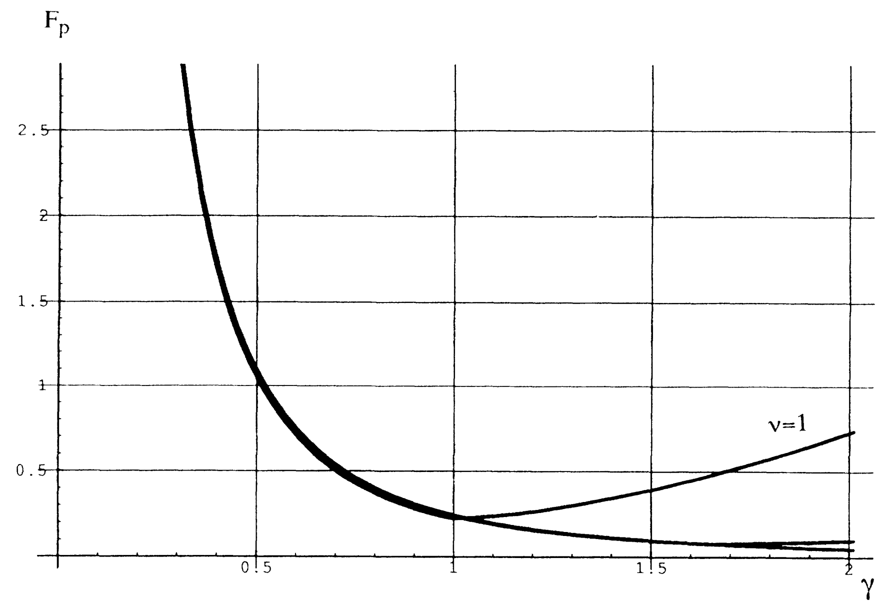

FIG. 6. F_p as function of γ for $\nu=1$, 0.6, 0.2 corresponding to curves from top to bottom.

if γ increases while the pre-exponent term increases with γ. Therefore, if d_0 is closed to one: $d_0>1$, and $\nu>1$, then the critical current j as a function of γ has a local maximum. Physically this optimum originates from competition between two factors: First, the minimum in the reflection at $\gamma\approx 1$ which reduces the order parameter (or enhances the proximity effect) and second is the reduction of ξ_n when γ grows, i.e., reduction in the tunneling amplitude.

The critical current also increases if γ becomes much less than one, an effect which is bigger for large ν. For small γ both factors mentioned above enhance j, i.e., both reflection and tunneling when $\gamma<1$ is reduced. Figure 7 shows the local maximum of $j(\gamma)$ as well as the enhancement at small γ for $d_0=1.5$ and $\nu=3$ [the ordinate is labeled by normalized value $j_0=jd_0/(47.7eT_cN_stv_s\tau^2)$].

We note that present theory is applicable close to T_c, i.e., for small τ. To estimate the range of applicability we note that the matching of the asymptotic of the solution [Eq. (21)] with the expansion of the Ginsburg-Landau solution Eq. (20) when δ_p is large, can be achieved if correspondingly τ is small. [We need to make such a match since the solution of integral equation (19) is defined up to the constant which is obtained from this matching.] Hence, the temperature interval where this theory is valid is given by

$$\tau\leq\frac{2}{(1+\delta_p)^2}\ . \tag{25a}$$

The same considerations are true in the case of junctions with large concentration of nonmagnetic impurities in the superconductors and in the weak link.

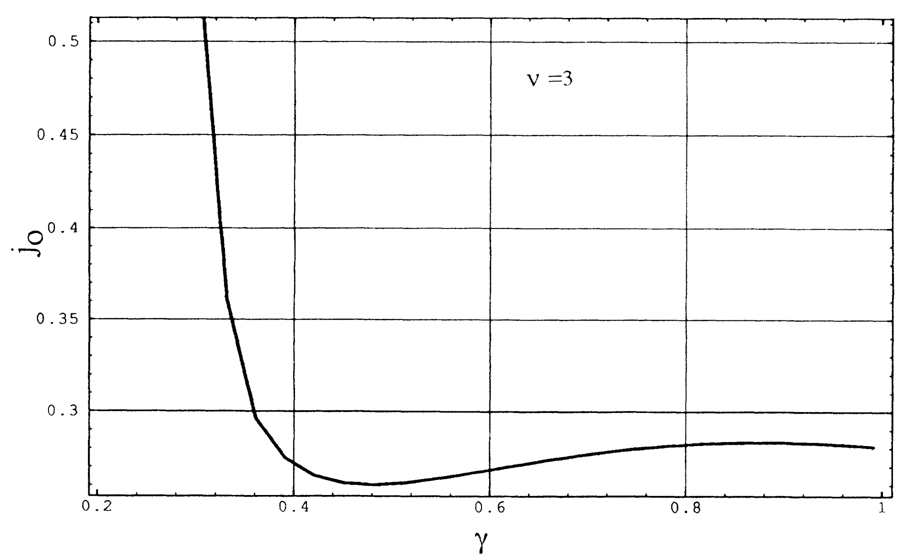

FIG. 7. The γ dependence of the critical current for $d_0=1.5$. Note the local maximum which is in the region of applicability of the theory ($d/\xi_n\sim 1.4$).

III. DIRTY SNS JUNCTION

In the dirty limit: $l_s < \xi_{si}$ and $l_n < \xi_{ni}$ (the index i stands for dirty limit). The relevant physical characteristics are the resistivities ρ_s and ρ_n, the densities of the quasiparticle states N_s and N_n (the latter are proportional to the Sommerfield constants χ_s and χ_n) for superconductive electrodes in the normal state and in the weak link, respectively. At temperatures near T_c, the density of the critical current was calculated by Barone and Ovchinnikov.[3] They numerically solved the integral equation (19) for dirty SNS junction and obtained the asymptotic form of the order parameter similar to Eq. (21). The parameter δ_i, which replaces δ_p for the dirty case, depends now on one quantity q

$$q=\frac{\rho_n\xi_n^*}{\rho_s\xi_S}\,,\quad \xi_n^*=\sqrt{D_n/2\pi T_c}=\xi_{si}\sqrt{\rho_s N_s/\rho_n N_n}\,, \tag{25b}$$

where $D_n=\frac{1}{3}v_n l_n$ is the diffusion coefficient of quasiparticles in the normal metal. Here we have obtained in the same way as in the pure limit (see Appendix A) a simple analytical expression for δ_i, which is in good agreement with the results of numerical computations:[3]

$$\delta_i=\frac{0.56q(1+0.81q)}{(1+q)}\ . \tag{26}$$

The amplitude of the Josephson critical current density j_i can also be represented in a simple analytical form. In this case the numerical data[3] for current density j_i has the form

$$j_i=\frac{1.86T_cF_i\tau^2}{e\xi_{si}\rho_s\sqrt{t}}=1.63\times10^{-4}\frac{\tau^2}{\sqrt{t}}F_i\frac{T_c(\mathrm{K})}{\xi_{si}\rho_s(\Omega\,\mathrm{cm}^2)}\ . \tag{27}$$

In this formula, the last term gives the current density in A/cm^2. For F_i we find

$$F_i=q(1-0.11\ln q)\exp\left[-\frac{d\sqrt{t}}{\xi_n^*}\right]\ . \tag{28}$$

For a given thickness $d\sim\xi_{ni}$ and fixed values of N_s and N_n, the critical current as a function of normal resistivity ρ_n has a maximum. This maximum, as in the clean limit, is due to competition between the suppression of the order parameter by the proximity effect (γ or $\rho_n/\rho_s\sim1$) and the reduction of the tunneling (exponential) (γ or $\rho_n/\rho_s\gg1$). This becomes clear if F_i is rewritten by introducing new combinations of the relevant parameters $\kappa^2=\rho_n/\rho_s$, $\varepsilon^2=N_s/N_n$:

$$F_i=\kappa\varepsilon[1-0.11\ln(\kappa\varepsilon)]\exp\left[-\frac{d\kappa\sqrt{t}}{\varepsilon\xi_{si}}\right]\ . \tag{29}$$

This maximum, however, appears in the range of thicknesses d which are on the margin of where the theory is applicable, $d\sim\xi_n$. The optimum thickness d_m is related to κ and ε by a simple relation: $\kappa d_m t^{0.5}/(\varepsilon\xi_{si})=0.89$. As an example, consider typical values for parameters of ceramic and noble metals with $\xi_{ni}\sim0.5$ nm, $\rho_s\sim2\times10^{-3}$ Ω cm, $\xi_n\sim40$ nm, and $N_s/N_n\sim6.4$. Assuming $d=\xi_n^*$ we get the optimum value $\rho_n=1.5\times10^{-6}$ Ω cm.

IV. PURE SNS JUNCTION ($T_{cn}\neq0$)

The Josephson current was obtained on experiments performed on a series of YBCO/$Y_{0.6}Pr_{0.4}$BCO/YBCO edge junctions.[17] The critical temperature of the normal layer (here, the $Y_{0.6}Pr_{0.4}$BCO compound) deviates from zero, $T_{cn}=40$ K. SNS junctions in this experiment belong to the clean case. For this limit a complete theory is not available. Here we are going to consider only the exponential factor in the expression for the critical current. For dirty SNS junctions the influence of the proximity effect on the exponent when $T_{cn}\neq0$ is well known;[18] instead of d/ξ_{ni}, we have dK_i where K_i is the inverse decay length and is given by the smallest root of the following equation:

$$\ln\left[\frac{T_{cn}}{T}\right]=\psi(\tfrac{1}{2}-\tfrac{1}{2}\xi_{ni}K_i)-\psi(\tfrac{1}{2})\ . \tag{30}$$

Here Ψ is the logarithmic derivative of the Γ function.

We can find the equation for pure limit similar to Eq. (30) in a manner analogous to Ref. 3. Let us consider an SN system and suppose that the normal layer occupies the half space $z<0$. The linear integral equation for order parameter [see Eq. (19) and Appendix A] has a form

$$\Delta(z<0)=-\frac{g_n}{v_n}N_n2\pi T\sum_{\omega>0}\int_0^1\frac{d\xi}{\xi}\int_{-\infty}^0 dz'\Delta(z')\exp\left[-2\frac{\omega}{\xi v_n}|z-z'|\right]\ . \tag{31}$$

This equation is valid only deep inside the normal metal where $\Delta(z)$ is the sum of damped exponential $\exp(K_pz)$. The smallest K_p gives the maximum contribution; thus we take $\Delta(z)=c\exp(K_pz)$, put it into Eq. (31), and perform the integration on z'. K_p is less than $\varepsilon_1=|\omega|/v_n$, and we can neglect the terms $\exp(2\varepsilon_1z)$ as compared with $\exp(K_pz)$. The result can be represented in the form

$$\ln\left[\frac{T}{T_{cn}}\right]=\frac{1}{2}\int_0^1 d\xi\left[2\psi\left[\frac{1}{2}\right]-\psi\left[\frac{1}{2}+\frac{K_pv_n\xi}{4\pi T}\right]-\psi\left[\frac{1}{2}-\frac{K_pv_n\xi}{4\pi T}\right]\right]\ . \tag{32}$$

After final integration over ξ, we obtain the equation which in the clean limit replaces Eq. (30),

$$\frac{1}{\Gamma(\frac{1}{2}+y)} = \left[\frac{7.12t}{b}\right]^y \left[\frac{\sin[\pi(\frac{1}{2}-y)]}{\pi}\right]^{1/2}, \tag{33}$$

where $y=K_p\xi_n/2$, $b=T_{cn}/T_c$. K_p is the smallest root of this equation. K_p enters in the expression for the critical current $j\sim\exp(-dK_p)$, a result which may be quite different from the dirty limit Eq. (30) for which $j\sim\exp(-dK_i)$.

V. PURE SNS JUNCTIONS AT LOW TEMPERATURE

For temperatures well below T_c the self-consistent determination of the order parameter $\Delta(z)$ and the critical current requires keeping high orders in $\Delta(z)$. However, in this case the model with abrupt pair potential[5,6,19] is physically reasonable and the calculation of the critical current is in fact easier. The most recent development of this model by including the barrier potential was given in Ref. 20 (see also Ref. 21). In this article we generalize this model by allowing distinct effective masses and Fermi velocities for the superconductors and the weak link. In the limit of wide link (see below) and for equal masses $m_s=m_n$ our result reduces to that of Ref. 20.

Here we present a brief scheme of calculations and single out the points which are connected with our generalization ($m_n\neq m_s$, $v_n\neq v_s$). We begin with rewriting Eq. (11) for the current density in the form ($\delta\to+0$)

$$J=\frac{e}{2im}(\nabla_{z'}-\nabla_z)T \times\sum_\omega\int\frac{d^2p}{(2\pi)^2}[e^{-i\omega\delta}G^s_{\omega p}(z,z')_{z\to z'}-\text{c.c.}] . \tag{34}$$

Equation (34) can be easily proven by using the Lehmann representation for the Green's function—the ω summation yields a zero contribution for the real part of $\exp(-i\omega\delta)G^s_{\omega p}(z,z')$.

The abrupt pair potential model is given by

$$\Delta(z)=\Delta,\quad |z|>d/2$$
$$=0,\quad |z|<d/2 . \tag{35}$$

The Green's function $G^s_{\omega p}(z,z')$, as well as the anomalous Green's function $F^+_{\omega p}(z,z')$ obey Gor'kov's equations

$$\left\{i\omega\sigma_3+I\left[\frac{mv^2}{2}-\frac{p^2}{2m}+\frac{1d}{2dz}\frac{1}{m}\frac{d}{dz}\right]+\Delta\begin{bmatrix}0 & f\\ -f^* & 0\end{bmatrix}\right\} \times\begin{bmatrix}G^s_{\omega p}(z,z')\\ F^+_{\omega p}(z,z')\end{bmatrix}=\begin{bmatrix}\delta(z-z')\\ 0\end{bmatrix}, \tag{36}$$

where f contains the phases ϕ_1, ϕ_2 [Eq. (12')], I and σ_3 are $2\otimes2$ diagonal matrices with elements [1,1], [1,−1], respectively.

The boundary conditions for $G^s_{\omega p}(z,z')$ are the same as for $G_{\omega p}(z,z')$ [see Eq. (2) and the preceding text]. The anomalous Green's function $F^+_{\omega p}(z,z')$ and $(1/m)\partial F^+_{\omega p}(z,z')/\partial z$ are continuous on z at each interface of the SNS junction. We also have

$$G^s_{\omega p}(z,z')_{z\to z'+0}=G^s_{\omega p}(z,z')_{z\to z'-0} ,$$
$$F^+_{\omega p}(z,z')_{z\to z'+0}=F^+_{\omega p}(z,z')_{z\to z'-0} , \tag{37}$$

and in addition to Eq. (2) there is a relation

$$\frac{d}{dz}F^+_{\omega p}(z,z')_{z\to z'+0}=\frac{d}{dz}F^+_{\omega p}(z,z')_{z\to z'-0} . \tag{38}$$

With these boundary conditions, we find the solution of Gor'kov's equations and evaluate the critical current. It is convenient to calculate j at the symmetry point $z=0$ (see Fig. 1), since at this point it is possible to avoid the appearance of a source term[20] which originates from the fact that $\Delta(z)$ does not satisfy the self-consistency equation. Therefore we need the Green's function $G^s_{\omega p}(z,z')$ at $z'\to z+0$, $z=0$. Since the order parameter does not depend on z inside the superconductors, there is a simple relation between the Green's functions $G^s_{\omega p}(z,z')$ and $F^+_{\omega p}(z,z')$ at $|z'|<d/2$,

$$G^s_{\omega p}(z,z')=\frac{\exp[i\phi_{1,2}]}{2m_s\Delta} \times\left[-2m_si\omega+(m_sv_s)^2\eta+\frac{d^2}{dz^2}\right]F^+_{\omega p}(z,z') , \tag{39}$$

where η is given by Eq. (9) and $z>d/2$ or $z<-d/2$ for ϕ_1,ϕ_2, respectively.

In the normal metal, $G^s_{\omega p}(z,z')$ and $F^+_{\omega p}(z,z')$ are connected only through the boundary conditions. However, we are seeking the Green's functions with $|z'|<d/2$ (in fact $z'\to0$). In different regions of z, they have the form

$$F^+_{\omega p}(z,z')=a_1e^{-iz\lambda_s}+a_2e^{iz\lambda_s^*} ,$$
$$G^s_{\omega p}(z,z')=\frac{-i\exp[i\phi_2]}{\Delta}(\omega_+a_1e^{-iz\lambda_s}-\omega_-a_2e^{iz\lambda_s^*}) ,\quad z<-d/2 ;$$
$$F^+_{\omega p}(z,z')=b_1e^{iz\lambda_s}+b_2e^{-iz\lambda_s^*} ,$$
$$G^s_{\omega p}(z,z')=\frac{-i\exp[i\phi_1]}{\Delta}(\omega+b_1e^{iz\lambda_s}-\omega_-b_2e^{-iz\lambda_s^*}) ,\quad z>d/2 ; \tag{40}$$
$$F^+_{\omega p}(z,z')=c_1e^{izk_n^*}+c_2e^{-izk_n^*} ,$$
$$G^s_{\omega p}(z,z')=d_1e^{izk_n}+d_2e^{-izk_n},\quad z<z',\quad |z|<d/2$$
$$=e_1e^{izk_n}+e_2e^{-izk_n},\quad z>z' .$$

All coefficients in Eq. (40) from a_1 to e_2 are functions of z'; $\omega_+=(\omega^2+\Delta^2)^{1/2}+\omega$, $\omega_-=(\omega^2+\Delta^2)^{0.5}-\omega$. From Eq. (20) and Eq. (37), if $|z|$, $|z'|<d/2$ we get

$$e_1=d_1+\frac{m_n}{ik_n}\exp[-ik_nz'] ,$$
$$e_2=d_2-\frac{m_n}{ik_n}\exp[ik_nz'] . \tag{41}$$

The wave numbers λ_s, k_n are obtained from the solution of Eq. (36) with the zero-valued right-hand side

$$\lambda_s=(p_s^2\eta+2\,\mathrm{i}m_s\sqrt{\omega^2+\Delta^2})^{0.5}\ , \tag{42}$$

where $k_n^2=(p_n^2\xi+2m_n i\omega)$; $p_n=m_n v_n$; $p_s=m_s v_s$, square roots are chosen here so that $\mathrm{Im}k_n>0$, $\mathrm{Im}\lambda_s>0$, and ξ,η are related by Eq. (9).

The continuity conditions at the interfaces $z=\pm d/2$ (there are eight equations) determine all the coefficients $(a,\ldots,d)$, hence $G^s_{\omega p}(z,z')$ for z,z' in the weak link is obtained. From Eq. (34) we derive the dc Josephson current

$$J=\frac{e\Delta^2}{\pi}m_n^2v_n^2\sin\phi T\sum_{\omega>0}\int_0^1 d\xi\Gamma_\omega^{-1}\ , \tag{43}$$

where

$$\Gamma_\omega=\Delta^2\cos\phi+[\omega^2+(\omega^2+\Delta^2)K^2]\cosh(d\tilde{k}_n)+2\omega\sqrt{\omega^2+\Delta^2}K_1\sinh(d\tilde{k}_n)$$
$$-(\omega^2+\Delta^2)K_2\cos(dk_n')+(\omega^2+\Delta^2)K_3\sin(dk_n')\ ,$$
$$K=\frac{|\lambda_s|^2+\nu^2|k_n|^2}{\nu|k_n|\lambda_s'},\quad K_1=\frac{|\lambda_s|^2+\nu^2|k_n|^2}{2\nu|k_n|^2\lambda_s'}k_n',\ K_2=\left[\frac{|\lambda_s|^2-\nu^2|k_n|^2}{\nu|k_n|\lambda_s'}\right]^2-\left[\frac{k_n'\tilde{\lambda}_s}{2|k_n|\lambda_s'}\right]^2\ , \tag{44}$$
$$K_3=\frac{|\lambda_s|^2-\nu^2|k_n|^2}{\nu|k_n|^2(\lambda_s')^2}k_n'\tilde{\lambda}_s,\ \lambda_s'=\lambda_s+\lambda_s^*,\quad k_n'=k_n+k_n^*,\quad \tilde{\lambda}_s=-i(\lambda_s-\lambda_s^*),\quad \tilde{k}_n=-i(k_n-k_n^*)\ .$$

We have maintained in Eq. (44) the term of order $(m_s v_s\xi_s)^{-1}$, while terms of order $(m_n v_n\xi_n)^{-1}$ are neglected. For high-T_c superconductors, the former terms have a noticeable influence in the case $\gamma\approx 1$, $\nu\approx 1$. If at a given temperature T, $d\gg\xi_n$, then the main contribution to the current density comes from integration over ξ near $\xi\approx 1$. The hyperbolic functions remain only in Γ_ω so that $K_n\approx m_n v_n$, $k_n'\approx 2k_n$, $K_1=K$, and

$$K^2=\left[\frac{\beta_1+\gamma^2}{2\gamma}\right]^2\frac{2}{1+\beta_1},\quad \beta_1=\left[1+\frac{4(\omega^2+\Delta^2)}{(m_s v_s^2)^2}\right]^{0.5}\ . \tag{45}$$

If the terms of order $(m_s v_s\xi_s)^{-1}$ are ignored then $\beta_1=1$ and

$$K=(1+\gamma^2)/2\gamma\ , \tag{46}$$

and the current has the form $J=j_a\sin\phi$, where

$$J=\frac{2e\Delta^2 m_n^2 v_n^3\sin\phi}{\pi^2 d[\pi T+K\sqrt{\Delta^2+(\pi T)^2}]^2}\exp\left[-\frac{2\pi T d}{v_n}\right]\ . \tag{47}$$

This expression reduces to that of Ref. 20 if $m_s=m_n$ when the barrier height is identified as $U=m_s v_s^2/2-m_n v_n^2/2$. Note, however, that for $d\le\xi_n$, a nontrivial dependence on $\nu=m_s/m_n$ appears.

If T is close to T_c, $j_a=Qj$ [with K represented by Eq. (46)] where Q is given by

$$Q=2\left[\frac{\Delta}{3.06\tau T_c(\delta_p++1.2+0.5a)}\right]^2\ , \tag{48}$$

j is given by Eq. (24), δ_p and a by Eq. (22) and $\tau=1-t$.

In the last formula let us substitute the order parameter Δ through its boundary value at $z=d/2$ [see Eq. (21) with $z=0$]. Then for the factor Q we have

$$Q=\left[\frac{\delta_p+a}{\delta_p+1.2+0.5a}\right]^2\ . \tag{49}$$

Note that the temperature interval near T_c where we have used the self-consistent theory (Sec. II) is rather small for large δ_p [see inequality (24a)]. However, when $\delta_p\gg 1$, Q is close to 1 and $j_a\approx j$, i.e., the model with abrupt pair potential can be extended to the region near T_c and Eq. (47) is valid for all $T<T_c$ with Δ substituted by its value at the SN boundary $\Delta(z=0)=\Delta_0(T)\tanh[(\tau/2)^{1/2}\delta_p]$. Thus at $T\ll T_c$, $\Delta(z=0)\approx\Delta_0(T)$ as in (47), while near T_c [Eq. (24a)] $\Delta(z=0)\sim\tau^{1/2}\Delta_0(T)$ is reduced by the proximity effect which affects the τ dependence of the critical current. The reason that Eq. (47) [with $\Delta\to\Delta(z=0)$] interpolates so well for all $T<T_c$ is that a large δ_p implies a weak z dependence [Eq. (21)] so that the abrupt pair potential is in fact valid even near T_c.

We note that Eq. (47) is a monotonically decreasing function of γ and the local maximum of Sec. II is absent. The local maximum is a result of a minimum in the reflection from the NS boundary which affects the $\Delta(z)$ solution. This effect is absent when $\Delta(z)$ is an abrupt pair potential [Eq. (35)].

VI. CONCLUSIONS

In this work we have studied SNS Josephson junctions which are necessary for analyzing weak links in composite high-T_c materials which are used for fabrication of wires and tapes. Our work generalizes previous results by allowing different Fermi velocities and effective masses in the superconductor and the weak link, respectively. We consider both clean and dirty limits, as well as temperatures near T_c or well below T_c.

An interesting result is Eqs. (47)–(49) which relate the self-consistent theory near T_c with a step-type approximation for the pair potential, which is a good approximation at temperatures well below T_c. When v_s/v_n deviates considerably from 1 [inequality (24a)] we find that both derivations coincide.

Our results allow for optimization of material parameters so that the critical current is maximized. In pure junctions the optimum is obtained by choosing ratios of Fermi velocities or ratios of effective masses. The critical current is maximized by high reflection from an SN boundary, i.e., γ far from 1, and by the tunneling factor $\exp(-d_0\gamma)$ [Eq. (24)]. The result of both factors is enhancement at $\gamma \ll 1$ and a local maximum at $\gamma \approx 1$ (see Fig. 7).

For dirty junctions the optimization is done by choosing the resistivity of the weak link. In this case the critical current is affected by the proximity effect (reducing j) which is reduced by a high resistivity ρ_n of the weak link and the exponential tunneling term which decreases with ρ_n [Eq. (21)], The result is a local maximum of $j(\rho_n)$ which can be used to optimize weak links in the dirty limit.

Finally, we consider experimental data[13] for Ag/YBCO SNS junctions which seem to be in the clean limit[13] in a large temperature range below T_c. The data are rather surprising, showing that $I_c(\tau)$ has a downward curvature which is opposite to that of the $\exp(-d/\xi_n)$ factor. A fit to the dirty-limit theory[2] was attempted, though an empirical dependence of the form

$$I_c = I_{c0}[a_0\tau + (1-a_0)\tau^2] \tag{50}$$

was found to be a better fit to the data[13] with $a_0 \approx 1.6$. We propose that Eq. (47) for the clean limit is more reliable for this experiment. Qualitatively, the factor K in Eq. (47) favors downward curvature (i.e., large a_0); we cannot, however, give a quantitative fit as there are uncertainties in the parameters γ and d/ξ_n. Furthermore, the rather low $T_c \approx 50$ K indicates strong disorder within the superconductors.

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APPENDIX A

In this appendix we derive the solution (21) for the self-consistency equation of $\Delta(z)$, Eq. (19). The kernel $K_{\omega p}(z,z')$ in the integral equation (19) is defined in the half-space occupied by the superconductor ($z>0$, $z'>0$). For Eq. (31) we need K_p in the interval $z<0$, $z'<0$. Therefore, we consider the Green's functions for all these ranges of z. From the solution of Eq. (1) for a SN two-layer system we find

$$G_{\omega p}(z,z') = -\frac{m_s}{k_s}[\exp(-k_s|z-z'|) - \alpha(\xi)\exp(-k_s(z+z'))], \quad z>0, \quad z'>0\ ;$$
$$-\frac{m_n}{k_n}[\exp(-k_n|z-z'|) + \alpha(\xi)\exp(k_n(z+z'))], \quad z<0, \quad z'<0\ .$$

If $z>0$, $z'<0$ or $z<0$, $z'>0$, expressions for the Green's functions can be easily obtained from the last two lines in Eq. (4). We ignore, as was mentioned above the oscillating terms in the kernel $K_{\omega p}(z,z')$ and obtain

$$K_{\omega p}(z,z') = \frac{m_s^2}{|k_s|^2}[\exp(-\tilde{k}_s|z-z'|) + |\alpha(\xi)|^2\exp(-\tilde{k}_s(z+z'))]\ . \tag{A1}$$

The normal metal has the same kernel as Eq. (A1), except that the index s is replaced by n and the minus in the second exponent is replaced by a plus. Equation (31), which we used in Sec. IV, neglects the term proportional to $\alpha(\xi)$ in the kernel. This term is small deep in the normal metal where we are seeking the solution of the integral equation.

After introducing (A1) into Eq. (19), the equation for the order parameter takes the form

$$\Delta(z) = |g|\frac{m_s^2}{2\pi}\sum_{n>0}\int_0^1 \frac{d\eta}{\eta}\int_0^\infty dz'\Delta(z')P(z,z')\ ,$$
$$P(z,z') = [\exp(-\tilde{k}_s|z-z'|) + |\alpha(\xi)|^2\exp(-\tilde{k}_s(z+z'))]\ , \tag{A2}$$

This is the Miln-type integral equation, and the solution can be found by the variational method proposed for Miln's equation.[22] First we write $\Delta(z)$ in the form similar to Eq. (21),

$$\Delta(z) = M\left[\frac{z}{\xi_{1s}} + S(z)\right], \quad S(z) = \delta_p + h(z) \tag{A3}$$

where the constant M is found from matching with the Ginzburg-Landau solution Eq. (20); it is equal to the coefficient before brackets on the right-hand side of Eq. (21). The function $h(z)$ goes to zero at $z \to \infty$. Substituting Eq. (A3) into the equation for the order parameter yields an equation for $S(z)$ and $h(z)$

$$h(z) = R(z) + \delta_p Q(z) + |g|\frac{m_s^2}{2\pi}\sum_{n=0}\int_0^1\frac{d\eta}{\eta}\int_0^\infty dz'h(z')P(z,z')\ ,$$
$$S(z) = R(z) + |g|\frac{m_s^2}{2\pi}\sum_{n=0}\int_0^1\frac{d\eta}{\eta}\int_0^\infty dz'S(z')P(z,z')\ , \tag{A4}$$
$$R(z) = |g|\frac{m_s^2}{8\pi^3 T}\frac{v_s^2}{\xi_{1s}}\sum_{n=0}\frac{1}{(2n+1)^2}\int_0^1 d\eta(1+|\alpha(\xi)|^2)\times\exp(-\tilde{k}_s z)\ ,$$
$$Q(z) = \frac{-|g|N_s}{2}\sum_{n=0}\frac{1}{(2n+1)}\int_0^1 d\eta\,\eta^{-0.5}(1-|\alpha(\xi)|^2)\times\exp(-\tilde{k}_s z)\ ,$$

where the terms with higher orders in τ were omitted. We multiply Eq. (A4) by z, integrate over $z>0$, and after simple calculations we obtain the equation

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\int_0^1 d\eta[1+|\alpha(\xi)|^2]\tilde{h}(\tilde{k}_s)=\delta_p\xi_s\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\int_0^1 d\eta\sqrt{\eta}[1-|\alpha(\xi)|^2]$$
$$-\xi_s\sqrt{12/7\zeta(3)}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\int_0^1 d\eta\eta[1+|\alpha(\xi)|^2]\ . \tag{A5}$$

Here $\zeta(3)\approx 1.2$ is the Riemann ζ function, $\tilde{h}(\tilde{k}_s)$ defines the Laplace transform [see Eq. (15)] of $h(z)$. This identity will be used further for the minimizing functional. Due to the relation $K_{\omega p}(z,z')=K_{\omega p}(z',z)$, the functional which after variation gives Eq. (A4), has a form[22]

$$H=\frac{\int_0^{\infty}dz\,S(z)\left[S(z)-|g|(m_s^2/2\pi)\sum_{n>0}\int_0^1(d\eta/\eta)\int_0^{\infty}dz'S(z')P(z,z')\right]}{\left[\int_0^{\infty}R(z)S(z)dz\right]^2}\ . \tag{A6}$$

On the optimum trajectory we have

$$H_{\min}^{-1}=\int_0^{\infty}R(z)S(z)dz\ . \tag{A7}$$

On the other hand, choosing the trial function as a constant, $S(z)=\text{const}$, we obtain from Eqs. (A6) and (A4)

$$H_{\min}=\frac{7\zeta(3)}{6|g|N_s\xi_s}\frac{\sum_{n=0}[1/(2n+1)^2]\int_0^1 d\eta(1-|\alpha(\xi)|^2)}{\left[\sum_{n=0}[1/(2n+1)^3]\int_0^1 d\eta\sqrt{\eta}(1+|\alpha(\xi)|^2)\right]^2}\ . \tag{A8}$$

With the help of Eq. (A5), we calculate the right-hand side of Eq. (A7):

$$H_{\min}^{-1}=\frac{|g|}{2}N_s\xi_s\sqrt{12/7\zeta(3)}\left[\frac{2}{3}\delta_p\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}-\sqrt{12/7\zeta(3)}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\int_0^1 d\eta\eta(1+|\alpha(\xi)|^2)\right]\ .$$

Inserting this expression into the left part of Eq. (A8), we have

$$\delta_p=0.85\left[\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\int_0^1 d\eta\eta[1+|\alpha(\xi)|^2]\right]+\frac{\left[\sum_{n=0}[1/(2n+1)^3]\int_0^1 d\eta\sqrt{\eta}[1+|\alpha(\xi)|^2]\right]^2}{\sum_{n=0}[1/(2n+1)^2]\int_0^1 d\eta[1-|\alpha(\xi)|^2]}\ .$$

Summation over n yields the result of Eq. (22) for δ_p with A_j, B_j given by Eq. (23). The correction to the asymptotic form of $\Delta(z)$ near the surface is found by taking the function $h(z)$ in the form of the last term in the brackets in Eq. (21). Introducing this $h(z)$ into Eq. (A5), we obtain the expression (22) for a.

In the dirty limit, the integral equation for the order parameter which substitutes Eq. (A2) was obtained in Ref. 3. The calculation of δ_i, which is similar to that of δ_p, was done in Ref. 8.

APPENDIX B

In this appendix we show that the function $\alpha(\xi)$ [Eq. (5)] is the reflection coefficient from a single SN boundary. Consider Eq. (1) for a single boundary, i.e., $m=m_n$ and $v=v_n$ for $z>0$, while $m=m_s$, $v=v_s$ for $z<0$. A plane-wave-type solution of the corresponding Schrödinger equation [i.e., without $\delta(z-z')$ in Eq. (1)] has the form

$$u(z)=\exp(k_s z),\quad z<0$$
$$=a\exp(k_n z)+b\exp(-k_n z),\quad z>0\ , \tag{B1}$$

so that $u(z\to-\infty)\to 0$; k_s, k_n are defined below Eq. (4). The boundary conditions are

$$u(+0)=u(-0)\ ,\qquad u'(+0)/m_n=u'(-0)/m_s\ , \tag{B2}$$

where $u'(z)=\partial u/\partial z$. Equations (B1) and (B2) yield

$$a=(m_n k_s+m_s k_n)/(2m_s k_n)\ ,\qquad b=(m_s k_n-m_n k_s)/(2m_s k_n)\ . \tag{B3}$$

The reflection coefficient from (B1) is then $|\alpha|=|b/a|$ in agreement with Eq. (5).

A simple estimate of α can be given by assuming that the dominant contribution of the reflection is from quasiparticles moving normal to the SN surface, i.e., $p=0$, and that at low temperatures compared with the Fermi energies, the $\omega(\sim T)$ term can be neglected. Thus $k_n \approx i m_n v_n$, $k_s \approx i m_s v_s$, and with $\gamma = v_s / v_n$

$$|\alpha| = \frac{|1-\gamma|}{1+\gamma} \tag{B4}$$

so that $|\alpha|$ has a minimum at $\gamma = 1$. This minimum is essential in the self-consistent theory of Sec. II as it enhances the proximity effect and reduces the critical current. This minimum leads to the minima in Figs. 2–7 and causes the local maximum in Fig. 7.

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