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Phase transition of the Aharonov-Bohm periodicity in metallic cylinders

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A scattering formalism is developed for a random potential on a cylinder with flux ϕ along its axis. Simulations show the possibility of a phase transition in the ϕ dependence of an ensemble-averaged conductance: The periodicity in ϕ changes from $h/2e$ to h/e as the potential is weakened. The critical potential decreases as the sample size increases such that the zero-flux conductivity is $\sim 150e^2/h$.

The study of electron interference in disordered medium has recently focused on the Aharonov-Bohm periodicity in the cylinder or ring geometry. The basic phenomenon[1] states that all measurable quantities on a cylinder must be periodic as a function of the flux along the cylinder axis with period h/e.

Interference effects in disordered systems were predicted[2] to yield a conductivity with period $h/2e$. This prediction was confirmed by experiments on Mg and Al cylinders,[3-5] two-dimensional grids,[6,7] and thin-film rings.[8] Small size rings, however, have shown h/e oscillations;[9] also the thin film rings[8] show a change in periodicity to h/e at high fields.

Theoretical developments have focused on the effects of ensemble averaging over a random potential. A "typical" conductivity as measured in an experiment is usually derived by averaging the conductivity over an ensemble of potentials.[2,10] It was predicted however that in small samples the ensemble average does not necessarily describe the measured quantity.[11] For example, in a one-dimensional ring electrons move along a single path and detect one fixed potential; the conductivity has then a period h/e. Small rings indeed have shown the period h/e.[9]

Numerical simulations on wide rings by Stone and Imry (SI)[12] have shown that if an average is taken over a sufficiently large ensemble a period $h/2e$ results. It was also shown that averaging the electron's energy is equivalent to ensemble average.[12] Thus in a small ring which has period h/e, increasing temperature would increase the energy average and thereby yield a period $h/2e$.

The effect of disorder on quantum interference was also studied by Nguyen, Spivac, and Shklovskii (NSS).[13] Based on simulations of a simplified model NSS claim that there is a phase transition from period h/e at low disorder concentration ($x<5\%$) to period $h/2e$ at higher concentrations ($x>5\%$). The NSS model involves a number of approximations. (a) The model considers the hopping limit, i.e., strong random potentials; hence, self-intersecting paths are neglected. Interference due to self-intersections can, however, be an essential feature. (b) The transmission amplitude is real. (c) Although the system is two dimensional, NSS assume that the external leads are one dimensional.

The NSS model was also studied within an effective medium theory;[14] an h/e period was found for all $x\neq 50\%$, while an $h/2e$ period appeared at the single point $x=50\%$. The symmetry of the potential distribution is also an important ingredient in the proof that an ensemble-averaged conductance has period $h/2e$.[10] Thus the nonsymmetric situation requires further study.

In the present work we test the NSS idea on a more realistic model which includes self-intersections, complex transmission amplitudes, and multichannel leads. We use a cylindrical geometry in which the average current flows parallel to the magnetic field as in the experiments of Refs. 3-5. Note that the NSS or SI geometries are somewhat different, i.e., they correspond to a wide ring in which the current flows perpendicular to the magnetic field. Following NSS we ensemble average the logarithm of the conductivity. We do not find the NSS-type transition; instead, however, we find a transition as function of the potential strength in an ensemble of finite systems. For random potentials $\pm v$ with $v<v_c$ the period is h/e while for $v>v_c$ it becomes $h/2e$. As the system size increases v_c becomes smaller and probably vanishes for infinite systems. If, however, the zero-flux conductivity $g(0)$ is fixed as the size increases, the transition is maintained at $g(0)\simeq 150e^2/h$. In the NSS limit $v\rightarrow\infty$ we expect a period $h/2e$ independent of x ($x\neq 0$). We finally comment on experimental realizations of the transition found here.

Consider a cylinder with a continuous coordinate z parallel to its axis and a discrete angle around the cylinder $\theta=2\pi m/N$, where m is an integer $0\leq m\leq N-1$ and N is the number of channels. The electron's wave function $\psi(z,m)$ satisfies the Schrödinger equation

$$\left\{-\frac{\partial^2}{\partial z^2}+\cos\left[\frac{2\pi}{N}\left(i\frac{\partial}{\partial\theta}-\phi\right)\right]+v(z,m)\right\}\psi(z,m)=E\psi(z,m), \quad (1)$$

where

$$\exp[(2\pi/N)\partial/\partial\theta]\psi(z,m)=\psi(z,m+1),$$

$v(z,m)$ is the random potential, ϕ is the flux in units of h/e, and E is the energy eigenvalue. Periodic boundary conditions are necessary in the θ direction, i.e., $\psi(z,m+N)=\psi(z,m)$. For $v(z,m)=0$ the eigenfunctions have a continuous wave vector q and a discrete one n (n is integer $0 \le n \le N-1$) so that

$$\psi_{q,n}(z,m) = \exp(iqz + 2\pi imn/N) ,$$

and eigenvalues $q^2 + \cos[2\pi(n+\phi)/N]$. For a given energy E, the wave vectors for the longitudinal motion are $q_n = \{E - \cos[2\pi(n+\phi)/N]\}^{1/2}$. We consider a random potential of the form $v(z,m) = \sum_i v_i \delta(z - z_i)\delta_{m,m_i}$. The v_i at N_p sites are chosen randomly so that a fraction x has $v_i = +v_0$ and a fraction $1-x$ has $v_i = -v_0$. The integral equation corresponding to (1) yields a set of N_p algebraic equations for the full wave function at the N_p sites,

$$\psi_{q_n,n}(i) = \exp(iq_n z_i + 2\pi inm_i/N) + \sum_j v_j \sum_{n'} (2iq_{n'}N)^{-1} \exp[iq_{n'}|z_i - z_j| + 2\pi in'(m_i - m_j)/N]\psi_{q_n,n}(j). \quad (2)$$

When the argument of $q_{n'}$ in the square root is negative, $\mathrm{Im} q_{n'} > 0$ and the corresponding channel does not have free incoming or outgoing waves. Equation (2) is solved by a straightforward inversion of an $N_p \times N_p$ matrix. This considerable simplification is due to the continuous spectrum q^2.

The amplitudes $t_{n'n}$ ($r_{n'n}$) for transmitting (reflecting) currents from channel n to channel n' are identified from the asymptotic behavior of $\psi_{q,n}(z,m)$ as $z \to \pm\infty$,

$$t_{n'n} = \delta_{n'n} + (2iN)^{-1}(q_n q_{n'})^{-1/2} \times \sum_i v_i \exp(-iq_{n'}z_i - 2\pi in'm_i/N)\psi_{q_n,n}(i),$$
$$r_{n'n} = (2iN)^{-1}(q_n q_{n'})^{-1/2} \times \sum_i v_i \exp(iq_{n'}z_i - 2\pi in'm_i/N)\psi_{q_n,n}(i). \quad (3)$$

The final ingredient is the multichannel Landauer formula[15] which gives the conductance (in units of e^2/h) as

$$g = \sum_{n,n'} |t_{n'n}|^2 \frac{2\sum_n (1/q_n)}{\sum_{n'}(1/q_{n'})(1 - \sum_n |t_{n'n}|^2 + \sum_n |r_{n'n}|^2)} . \quad (4)$$

In the limit $N \to \infty$ g reduces to the simplified form[15] $g^s = \sum_{n,n'} |t_{n'n}|^2$. By unitarity and time reversal it can be shown that $g^s(\phi) = g^s(-\phi)$. Data on small rings[9] show, however, an asymmetry with respect to field reversal; we therefore use Eq. (4) as the appropriate finite system value.

The conductance exhibits fluctuations which depend on the choice of random potential and on the energy E.[16] These fluctuations are enhanced by numerical limitations on the sample size and number of samples in the ensemble average. To minimize the latter fluctuations we simultaneously average on the potential and on the energy; both averages being equivalent as argued by SI. Note also that Eq. (4) has discontinuities at energies

$$E = \cos[2\pi(n+\phi)/N] \quad (n = 1, 2, \ldots, N) ,$$

where the number of scattering channels changes.[15] The energy average smooths these discontinuities. In our simulations we choose the energy randomly in the range $E = 0.2 \pm 0.05$.

We present here ensemble averages on $\log_{10}[g(\phi)]$. The distribution of $\log_{10}(g)$ is considered to be closer to a normal distribution and the average $\langle \log_{10}(g) \rangle$ is then a better representation of a "typical" conductance.[13] Note also that the system size for coherent elastic scattering [as in Eq. (1)] is limited by a thermal inelastic scattering length l_i. Experimentally,[3,5] cylinder lengths are $\sim 10^4 l_i$ while the diameter is $\lesssim l_i$. Thus experiment measures an average on many subsystems whose size L is of order l_i.

We choose z_i for convenience as integers $1 \le z_i \le L$ so that the scattering centers are on a lattice of size $N_p = N \times L$. We check each solution for unitarity and average over I samples in an ensemble.

Figure 1 shows two distinct ϕ dependences. For $v_0 = 0.2$ we find a behavior, as found by SI, consistent with a period $h/2e$. The small component with period h/e is within the fluctuation range, which for $\log_{10}[g(\phi)/g(0)]$ is ~ 0.1. For $v_0 = 0.095$ we find a clear component with period h/e, i.e., $h/2e$ is not a period. The distinct values at $\phi = \frac{1}{2}$ and $\phi = 0$ remain so even for $I = 500$, so that the SI claim does not hold for a small v_0; the period $h/2e$ does not emerge with increasing ensemble size.

We consider $\gamma \equiv \langle \log_{10}[g(\frac{1}{2})/g(0)] \rangle$ as an order parameter; when it is zero (within ~ 0.1) the period is $h/2e$, and when it is distinctly nonzero the period is h/e. Figure 2 shows this order parameter for 10 ($=N$) channels, various L and I. The 10×20 system shows a clear deviation from period $\frac{1}{2}$ below a critical $v_0 = v_c$ with $v_c \simeq 0.15$; for $I = 50$ irregular fluctuations in the v_0 dependence (of order ~ 0.1) are still apparent while for $I = 100$ the v_0 dependence is smooth. Increasing size to 10×30 or 10×40 shifts the transition to a lower v_c, $v_c \simeq 0.13$. In the infinite-size limit the perturbation calculation[2] should be valid so that $v_c \to 0$.

Figure 3 shows γ as a function of $\langle \log_{10}[g(0)] \rangle$ rather than v_0; the data correspond to the range $0.06 \le v_0 \le 0.27$. For $v_0 > 0.27$, γ stays near zero but $\langle \log_{10}[g(0)] \rangle$ stops increasing, with values fluctuating near 1.9; this indicates an increasing role of fluctuations as v_0 increases. Furthermore, as v_0 increases the amplitude of the ϕ dependence

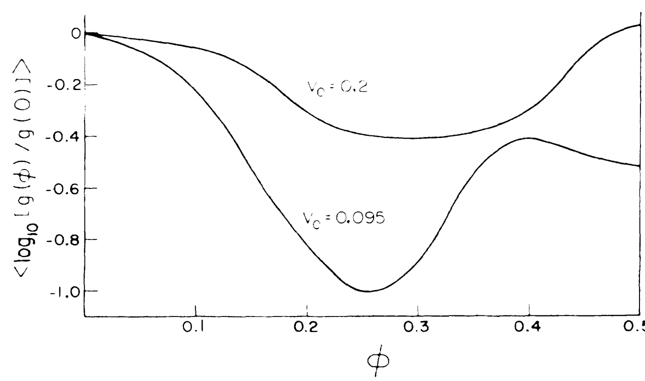

FIG. 1. Flux dependence for 75 samples of size 10×20 and $x = 4\%$. (Flux is $h\phi/e$.)

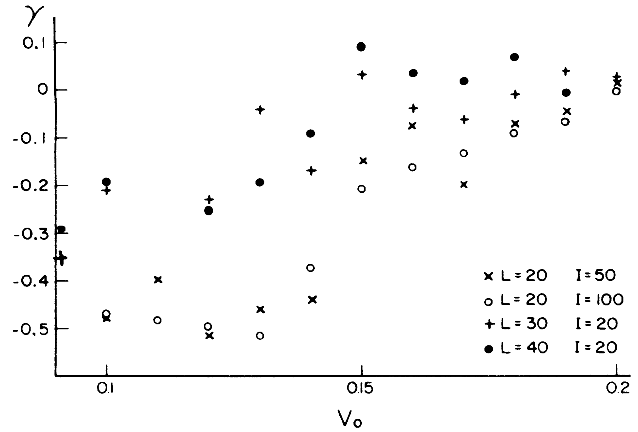

FIG. 2. Potential dependence of $\gamma = \langle \log_{10}[g(\frac{1}{2})/g(0)] \rangle$ with average on I samples of size $10 \times L$ and $x = 4\%$.

decreases below the 0.1 fluctuation range, a trend seen in Fig. 1.

Figure 3 shows two remarkable features. First, most of the irregular dependence of γ on v_0 (Fig. 2) is smoothed in Fig. 3. Thus γ correlates directly with $\langle \log_{10}[g(0)] \rangle$ and the fluctuations in Fig. 2 are due mainly to an irregular dependence of $\langle \log_{10}[g(0)] \rangle$ on v_0. Second, the transition into period h/e is around $\langle \log_{10}[g(0)] \rangle \simeq 2.2$ and is independent of L. Thus the larger systems have a lower v_c, but since the conductance decreases with size the transition occurs at the same typical conductance of $g(0) \simeq 150$. This implies a transition even in the infinite-size system, provided a limit $v_0 \rightarrow 0$ is taken such that $g(0)$ remains finite. Note also that the simulations by SI have $g(0) \simeq 3$, which is indeed in the period $h/2e$ regime of Fig. 3.

Our numerical accuracy is not sufficient to claim a strict phase transition. The SI data show however that γ converges to zero for large v_0 [or small $g(0)$] while our data definitely show a nonzero γ for small v_0 [or large $g(0)$]. The $\gamma \rightarrow 0$ situation may result when $I \rightarrow \infty$ even for finite systems.[12] If so, $\gamma(v_0)$ is nonanalytic and a phase transition in a finite, though averaged, system is implied. If, however, finite-size corrections maintain $\gamma \neq 0$ even as $I \rightarrow \infty$, then a strict phase transition is suggested only in infinite samples with the above $v_0 \rightarrow 0$ limit.

We find that γ depends rather weakly on the concentration x. This relates to the much weaker dependence of $\log_{10}[g(0)]$ on x than on v_0. In the NSS limit $v_0 \rightarrow \infty$ $g(0) \rightarrow 0$ and we expect a period $h/2e$ for any finite x. Thus, the NSS transition does not survive in our more realistic model. Nevertheless, our transition is similar to that of NSS since in both cases it is the increasing effect of the random potential which induces the period $h/2e$.

The transition found here can be interpreted as defining a mean free path l. When $l < L$, scattering is significant and diffusive motion yields the $h/2e$ period; if $l > L$, ballistic motion dominates and yields the h/e period. Since l is determined by the potential parameters v and x, our result implies that the conductance $g(l,L)$ is ~ 150 when $l(v,x) \simeq L$. This interpretation implies that the transition in a finite system, though fairly sharp, is not strictly a phase transition.

Finally, we comment on two types of experiments which can be sensitive to the transition found here. Consider first data on thin film rings[8] which show an h/e period at high magnetic fields while an $h/2e$ period is present at low fields together with an uncertain amount of an h/e component. In this geometry the magnetic field penetrates the ring itself, and at high fields the flux enclosed by different trajectories around the ring can differ by more than h/e. The logarithmic divergence associated with localization[2] is then suppressed by a new cutoff[8,17] and the conductance is enhanced. If this cutoff has the effect of reducing L, a transition from period $h/2e$ to period h/e with increasing field is possible. The h/e component then increases with field rather than staying constant as argued by SI. Further simulations on wide rings are needed to confirm this scenario.

Second, note that by increasing temperature $l_i (\simeq L)$ is decreased and a transition into period h/e is possible. This competes with an opposite tendency of increasing energy average which reduces the h/e component.[12] For large ensembles the first effect should dominate, i.e., period h/e appearing at higher temperatures.

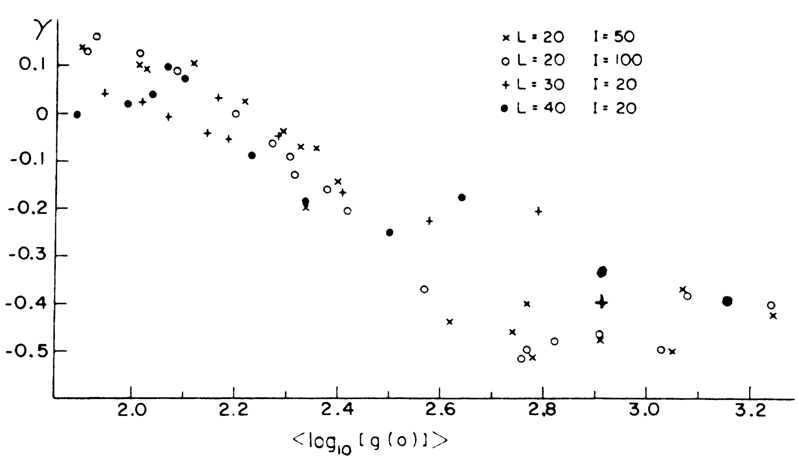

FIG. 3. As in Fig. 2 for the dependence on $\langle \log_{10}[g(0)] \rangle$.

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