Transconducting Transition for a Dynamic Boundary Coupled to Several Luttinger Liquids

B. Horovitz
Department of Physics, Ben Gurion University, Beer Sheva 84105, Israel

T. Giamarchi
DPMC-MaNEP, University of Geneva, 24 Quai Ernest Ansermet, 1211 Geneva 4, Switzerland

P. Le Doussal
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, PSL University CNRS, Sorbonne Universités, 24 rue Lhomond, 75231 Paris Cedex 05, France

(Received 6 May 2018; published 17 October 2018)

We study a dynamic boundary, e.g., a mobile impurity, coupled to N independent Tomonaga-Luttinger liquids (TLLs) each with interaction parameter K. We demonstrate that for $N \geq 2$ there is a quantum phase transition at $K \geq 1/2$, where the TLL phases lock together at the particle position, resulting in a nonzero transconductance equal to e^2/Nh. The transition line terminates for strong coupling at $K = 1 - (1/N)$, consistent with results at large N. Another type of a dynamic boundary is a superconducting (or a Bose-Einstein condensate) grain coupled to $N \geq 2$ TLLs; here the transition signals also the onset of a relevant Josephson coupling.

DOI: 10.1103/PhysRevLett.121.166803

There is considerable interest in systems of N independent one-dimensional Tomonaga-Luttinger liquids (TLLs) coupled via a dynamic boundary, e.g., a mobile impurity. This case has been realized in cold atom experiments employing a variety of impurity atoms, in either boson or fermion systems [1–11]. These studies range from polaronic effects in bulk baths [1–3], the approach to equilibrium [4–7] and more recently to one-dimensional cold atom gases [8–11]. Quantum impurities in TLL have been extensively studied [12–20], focusing on the particle dynamics and response to an external force.

A second type of dynamic boundary is realized by a superconducting grain, or a Bose-Einstein condensate (BEC), illustrated in the left panel of Fig. 1. This can be realized with wires formed on $LaAlO_3/SrTiO_3$ nanostructures [21] or with carbon nanotubes. The latter system [22] has in fact shown surprisingly large values of supercurrents. Of further interest are topological superconductors with Majorana islands coupled to TLLs via multiterminals. Theoretical studies [23–25] show the phenomena of interterminal conductance, with possible realizations in various experimental setups [26,27]. In dynamically coupled TLLs, as we show here, an analogous phenomenon, transconductance (see below), can occur even without Majorana states.

The case of an infinite number of TLLs was previously examined [28] and showed a phase transition in which the impurity can localize for a repulsive TLL. Understanding the finite N case is thus important in view of the experimental realizations [22,26,27] and the theoretical studies [23–25]. Studies of drag conductance in crossed carbon nanotubes [29–34] provide further motivation for studying $N > 1$.

In this Letter, using the renormalization group (RG) and duality methods akin to the study of quantum Brownian motion in a periodic lattice [35,36] we solve the problem of an impurity coupled to N TLLs. We demonstrate that there is a quantum phase transition for any $N \geq 2$ and show that the order parameter of this transition is provided by the transconductance, i.e., driving a current in one chain by applying a voltage on another chain. The transconductance vanishes in one phase and is e^2/Nh in the strong coupled phase. We show that the phase boundary interpolates between $K = 1/2$ at weak particle-TLL coupling and $K = 1 - (1/N)$ at strong coupling and discuss the experimental consequences.

We start with the dynamic impurity problem, and eventually consider the equivalent superconducting grain system. We focus on equilibrium dynamics with imaginary time τ. The particle position is denoted by X_τ and its motion is described by the action

$$S_{\mathrm{imp}} = \int d\tau 1/2M_0 \dot{X}_\tau^2 - g_0 \sum_{i=1}^{N} \rho_i(X_\tau, \tau), \qquad (1)$$

which includes the kinetic energy and identical instantaneous contact interactions with the densities $\rho_i(x, \tau)$, $i = 1, \ldots, N$ of the TLLs. The density of each TLL entering

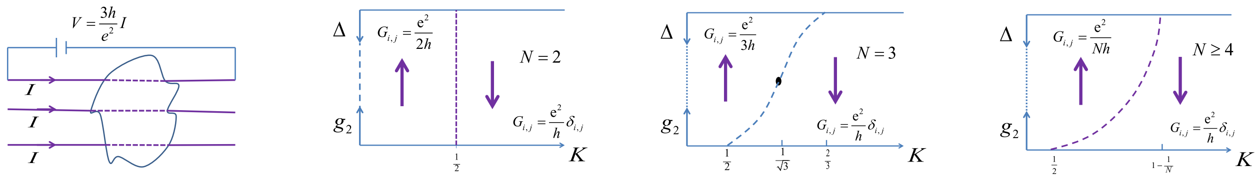

FIG. 1. Setup and phase diagrams. The left frame shows a setup with a superconducting grain Josephson coupled to $N = 3$ TLLs showing transconductance measurement in the coupled phase. The phase diagrams show g_2 or Δ ($\ln \Delta \sim -\sqrt{g_2}$ for large g_2) versus the TLL interaction parameter K for $N = 2, 3, \geq 4$, respectively, in a mobile impurity system; for a superconducting grain, replace K by $1/(2K_\rho)$ where K_ρ is the TLL parameter in the charge sector; $K, K_\rho < 1 (> 1)$ correspond to repulsive (attractive) TLL interactions. Dashed lines are phase transition lines (lines of RG fixed points) separating a decoupled phase where $g_2 \to 0$ is irrelevant and a coupled phase with strong transconductance where $g_2 \to \infty$ (equivalently $\Delta \to 0$); RG flow directions are indicated by thick purple arrows (they are vertical since K does not flow). The conductance matrices in the two phases are shown (add prefactor 2 for the spinfull superconducting grain). The conductance matrix varies continuously along the phase transition lines. The dot on the $N = 3$ case corresponds to a self-dual point at $K = 1/\sqrt{3}$, see Eq. (24).

in the coupling (1), is related to its standard phase field [37] $\phi_i(x, \tau)$ as

$$\rho_i(x, \tau) = \rho_0 + \alpha_1 \rho_0 \cos[2\pi\rho_0 x - 2\phi_i(x, \tau)], \tag{2}$$

where ρ_0 is the average density of each TLL and α_1 is nonuniversal. Inserted in (1) it leads to a direct coupling of the particle position to the oscillating part of the TLL density, which is the important effect here. We have neglected in (2) higher harmonics of the density, as well as the slowly varying part $-\partial_x \phi_i(x, \tau)/\pi$ of the density, which leads to subdominant corrections to the particle motion at low frequencies. The total action of the system is $S = S_{\mathrm{TLL}} + S_{\mathrm{imp}}$, where S_{TLL} is the standard action for N independent TLLs with common Luttinger parameter K

$$S_{\mathrm{TLL}} = \sum_{i=1}^{N} \int dx d\tau \frac{1}{2\pi K} [(\partial_\tau \phi_i)^2 + (\partial_x \phi_i)^2] \tag{3}$$

and we work in units such that the phonon velocity $u = 1$. Let us express the position X_τ in units of $(2\pi\rho_0)^{-1}$. The action of the particle becomes

$$S_{\mathrm{imp}} = 1/2M \int_\omega \omega^2 |X_\omega|^2 - g \int_\tau \sum_{i=1}^{N} \cos[X_\tau - 2\phi_i(X_\tau, \tau)], \tag{4}$$

where $M = M_0(2\pi\rho_0)^{-2}$, $g = \alpha_1 \rho_0 g_0$, and here and below $\int_\omega f(\omega) \equiv (1/\beta)\sum_{\omega_n} f(\omega_n) \overset{\beta\to\infty}{\to} \int (d\omega/2\pi)$ with $\int_\tau = \int_0^\beta d\tau$. We focus on the zero temperature $T = 1/\beta \to 0$ limit unless stated otherwise.

The theory defined by (3) and (4) is still highly nonlinear in the particle position X_τ and difficult to treat exactly. However, we claim that an equivalent long time theory is obtained by

$$\cos[X_\tau - 2\phi_i(X_\tau, \tau)] \to \cos[X_\tau - 2\phi_i(0, \tau)]. \tag{5}$$

This amounts to assuming that the TLL correlations are dominated by time differences $|\tau - \tau'| \gg |X_\tau - X_{\tau'}|$, which is satisfied by the X_τ correlations that we find below. Since only the term $\phi_i(0, \tau)$ appears in the coupling, we can integrate the Bose fields at all other points $\phi_i(x \neq 0, \tau)$, leading to the well studied $\sim|\omega|$ term [37]; hence the total action becomes

$$S = \frac{1}{2} \int_\omega \left(M\omega^2 |X_\omega|^2 + \sum_{i=1}^{N} \frac{2|\omega|}{\pi K} |\phi_\omega^i|^2 \right) - g \int_\tau \sum_{i=1}^{N} \cos(X_\tau - 2\phi_\tau^i), \tag{6}$$

where $\phi_i(0, \tau) = \phi_\tau^i$ and its Fourier transform is ϕ_ω^i. We now denote $B_\tau^i = 2\phi_\tau^i - X_\tau$ as the fields entering in the nonlinear term and define $\tilde{X}_\omega$ in Fourier via

$$X_\omega = \tilde{X}_\omega - \frac{1}{N_\omega} \sum_{i=1}^{N} B_\omega^i, \qquad N_\omega = N + 2\pi M K |\omega|, \tag{7}$$

where N_ω can be thought as an effective number of degrees of freedom. It is then easy to see that the action (6) can be rewritten as a sum over two independent sectors, the field $\tilde{X}$ on one hand, and the B_i's on the other, as

$$\begin{aligned} S &= \frac{1}{2} \int_\omega \left\{ \frac{|\omega|}{2\pi K} N_\omega |\tilde{X}_\omega|^2 + D_{i,j}^{-1} B_\omega^i B_\omega^{j*} \right\} \\ &\quad - g \sum_{i=1}^{N} \int_\tau \cos B_\tau^i, \\ D_{i,j}^{-1} &= \frac{|\omega|}{2\pi K} \left(\delta_{i,j} - \frac{1}{N_\omega} \right), \\ D_{i,j} &= \frac{1}{M\omega^2} + \frac{2\pi K}{|\omega|} \delta_{i,j}. \end{aligned} \tag{8}$$

Hence, one can first study the problem defined by the B_i fields, and in a second stage obtain the position of the particle X_τ from (7) as the sum of two independent terms. This decomposition immediately leads to two exact bounds, first [38]

$$\langle |X_\omega|^2\rangle \geq \langle |\tilde{X}_\omega|^2\rangle = \frac{2\pi K}{\omega N_\omega}, \qquad (9)$$

where $\langle \ldots \rangle$ denotes the average over the action S. Furthermore

$$\langle \cos X_\tau\rangle \lesssim \langle \cos \tilde{X}_\tau\rangle \sim \left(\frac{4\pi^2 KM}{N\beta}\right)^{K/N} \overset{\beta\to\infty}{\to} 0. \qquad (10)$$

Hence, the finite N behavior of $\langle \cos X_\tau\rangle$ differs from the $N\to\infty$ case [28] where it can be finite and then serve as an order parameter.

To 0th order in g, with $\langle \ldots \rangle_0$ denoting an average with respect to $S_{g=0}$,

$$\langle \cos B^i_\tau\rangle_0 = e^{-1/2\int_\omega \{(2\pi K/|\omega|)+(1/M\omega^2)\}} = 0, \qquad (11)$$

which is *strongly irrelevant* and cannot lead to an ordering of each individual B^i_τ. Naively, one could conclude from power counting that the coupling g is washed away by fluctuations, leading effectively to the $g=0$ theory.

However, this is not the case, as we have found: although strongly irrelevant, the terms $g\cos B^i_\tau$ generate an effective coupling $g_2\cos(B^i_\tau - B^j_\tau)$ between pairs of distinct fields. Indeed, the effective action evaluated to second order in perturbation theory in g contains a term $\cos(B^i_\tau - B^j_{\tau'})$ multiplied by

$$\begin{aligned} g^2\langle e^{iB^i_\tau - iB^j_{\tau'}}\rangle_0^{i\neq j} &= g^2 e^{-(1/2M)|\tau-\tau'| - \int_\omega (2\pi K/|\omega|)} \\ &\to \delta(\tau-\tau')Mg^2 e^{-\int_\omega (2\pi K/|\omega|)} \end{aligned} \qquad (12)$$

and integrated over times. We note that the finite mass is crucial to provide a short time cutoff $\sim M$. The action involving the B^i_ω fields (denoted as $\mathbf{B}_\omega$) can thus be replaced by the effective action

$$S_1 = \frac{1}{2}\int_\omega D^{-1}_{i,j} B^i_\omega B^{j*}_\omega - g_2\Lambda \sum_{\mathbf{V}}\int_\tau e^{i\mathbf{V}\cdot\mathbf{B}_\tau}, \qquad (13)$$

where $g_2\Lambda \sim Mg^2$, g_2 is a running dimensionless coupling, and Λ a high frequency cutoff with initial value $\sim M$. The vectors $\mathbf{V}$ are N dimensional, have one entry of $+1$, one of -1, and all other entries are 0, i.e., $\mathbf{V}\cdot\mathbf{B}_\tau = B^i_\tau - B^j_\tau$ with $i\neq j$. Hence, $\mathbf{V}$ form the primitive unit cell of an $N-1$ dimensional lattice that is perpendicular to the vector (1,1,...,1) on a simple cubic N dimensional lattice. This type of model appears in various contexts, e.g., the quantum Brownian motion in a periodic potential [35,36]. To second order the RG flow equation is (see [39])

$$\Lambda\partial_\Lambda g_2 = (1-2K)g_2 + \alpha(N-2)g_2^2 + O(g_2^3), \qquad (14)$$

where $\alpha = O(1) > 0$ is nonuniversal, and depends on a smooth cutoff procedure. Note that the TLL parameter K is not renormalized and does not flow. From (14) there is clearly a critical line for $K > 1/2$ and $N > 2$

$$g_2^c = \frac{2[K-(1/2)]}{\alpha(N-2)} \qquad (15)$$

such that for $g_2 < g_2^c$ the RG flow is towards the Gaussian $g_2 = 0$ theory, while for $g_2 > g_2^c$, g_2 flows to strong coupling, signaling a phase where the relative fields B^i_τ lock together, in a way that we study below.

The $N=2$ case has a single nonlinear term $\sim\cos(B^1_\tau - B^2_\tau)$, equivalent to the static impurity problem [40,41] and has a vertical phase boundary at $K=1/2$ (Fig. 1). Going back to general N, we now study the B^i_τ correlations by adding a source term $-\int_\omega |\omega|\mathbf{B}_\omega\cdot\mathbf{A}_{-\omega}$ to the action (13) so that $\langle B^i_\omega B^j_{-\omega}\rangle = (1/Z_1\omega^2)(\delta^2 Z_1/\delta A^i_{-\omega}\delta A^j_\omega)|_{\mathbf{A}=0}$, where [38] $Z_1 = \int \mathcal{D}\mathbf{B} e^{-S_1}$ is the partition sum in presence of the source. Before studying the general correlations, we note [39] an exact sum rule of the effective model (13) $\sum_i \langle B^i_\omega B^j_{-\omega}\rangle = N_\omega/(M\omega^2)$ for each j.

We proceed to study the strong coupling fixed point by a duality transformation. The process is well known for the $N=2$ case [37,40], results are also stated for the quantum Brownian motion [35,36], yet the extension to $N>2$ of our case involves some subtleties. We perform first a change of variables so that the Gaussian part of S_1, Eq. (13), becomes diagonal,

$$C^i_\omega = B^i_\omega - \alpha_\omega \bar{B}_\omega, \qquad \alpha_\omega = 1-\sqrt{1-N/N_\omega}, \qquad (16)$$

where $\bar{B}_\omega = \sum_i B^i_\omega/N$ and $\bar{C}_\omega = \sum_i C^i_\omega/N = (1-\alpha_\omega)\bar{B}_\omega$. The action becomes

$$\begin{aligned} S_1 &= 1/2\int_\omega \frac{|\omega|}{2\pi K}\mathbf{C}_\omega\cdot\mathbf{C}_{-\omega} - g_2\Lambda\sum_{\mathbf{V}}\int_\tau e^{i\mathbf{V}\cdot\mathbf{C}_\tau} \\ &\quad - \int_\omega |\omega|\left[\mathbf{C}_\omega\cdot\mathbf{A}_{-\omega} + \frac{\alpha_\omega}{1-\alpha_\omega}\bar{C}_\omega\sum_i A^i_{-\omega}\right]. \end{aligned} \qquad (17)$$

We consider next large g_2 where the trajectories of $\mathbf{C}_\tau$ are dominated by instantons, i.e., a sequence of n sharp jumps at consecutive times $\tau_1, \tau_2, \ldots, \tau_\alpha, \ldots, \tau_n$. The instantons shift $\mathbf{C}_\tau$ between neighboring equivalent minima of the g_2 term by vectors chosen from a set $\mathbf{R}_i$ such that $\mathbf{R}_i\cdot\mathbf{V}_j = \delta_{i,j}$. Hence, $\mathbf{R}_i$ form the reciprocal lattice to $\mathbf{V}_j$, each vector has one entry of $-1+1/N$ and all the rest are $1/N$, with norm $|\mathbf{R}_i|^2 = 1-1/N$. The vectors $\mathbf{R}_i$ are also orthogonal to (1,1,1,...), however, they do not form a primitive unit cell for $N>3$ and then their lattice symmetry differs from that of the $\mathbf{V}_i$. For example, for $N = 3$ both $\mathbf{V}_i$, $\mathbf{R}_j$ form a 2D triangular lattice; however, for $N = 4$ there are 12 vectors $\mathbf{V}_i$ forming an fcc lattice while there are 8 vectors $\mathbf{R}_i$ that form a bcc lattice.

Since $\mathbf{R}_\alpha$ are perpendicular to (1,1,1,...) instanton trajectories do not describe the center of mass $\bar{C}_\omega$. Decoupling this center of mass is achieved by the shift $\tilde{C}^i_\omega = C^i_\omega - \bar{C}_\omega$; hence the Gaussian part in Eq. (17) decouples into $\mathbf{C}_\omega \cdot \mathbf{C}_{-\omega} = \tilde{\mathbf{C}}_\omega \cdot \tilde{\mathbf{C}}_{-\omega} + N|\bar{C}_\omega|^2$. The $\tilde{\mathbf{C}}_\tau$ trajectory is described by $\tilde{\mathbf{C}}(\tau) = 2\pi \sum_\alpha \mathbf{R}_\alpha \theta(\tau - \tau_\alpha)$. The coupling with the source can be written as $\int_\omega |\omega| \tilde{\mathbf{C}}^i_\omega \cdot \mathbf{A}^i_{-\omega} = i2\pi \mathbf{R}_\alpha \cdot \mathbf{a}(\tau_\alpha)$ where $\mathbf{a}_\omega = -\mathrm{sign}\omega \mathbf{A}_\omega$. The weight of each instanton is defined as $\Lambda\Delta \sim e^{-S_{\rm ins}}$ where [37,40] $S_{\rm ins} \sim \sqrt{g_2}$. In the strong coupling limit $\Delta = 0$, instantons are absent and the correlations become

$$\langle B^i_\omega B^j_{-\omega} \rangle = \frac{1}{(1-\alpha_\omega)^2} \langle |\bar{C}_\omega|^2 \rangle = \frac{N_\omega}{NM\omega^2} \qquad (18)$$

so that all TLLs becomes equally coupled to each other. If $\Delta > 0$ the term $1/2 \int_\omega (|\omega|/2\pi K) \tilde{\mathbf{C}}_\omega \cdot \tilde{\mathbf{C}}_{-\omega}$ produces, after integration on ω, logarithmic interactions between instantons [39] which correspond to the dual action,

$$S_2 = 1/2 \int_\omega \frac{K|\omega|}{2\pi} |\boldsymbol{\theta}(\omega)|^2 - \Lambda\Delta \sum_{\mathbf{R}} \int_\tau e^{i\mathbf{R}\cdot[\boldsymbol{\theta}(\tau) + 2\pi \mathbf{a}(\tau)]}. \qquad (19)$$

By shifting $\boldsymbol{\theta} \to \boldsymbol{\theta}(\tau) - 2\pi\mathbf{a}(\tau)$ and taking a second derivative in $\mathbf{A}_\tau$ we obtain a relation between the $\mathbf{B}_\omega$ and $\boldsymbol{\theta}_\omega$ correlations

$$\langle B^i_\omega B^j_{-\omega} \rangle = \frac{N_\omega}{NM\omega^2} + \frac{2\pi K}{|\omega|} \delta_{i,j} - K^2 \langle \theta^i_\omega \theta^j_{-\omega} \rangle. \qquad (20)$$

The dual form allows for deriving the RG equation, using $|\mathbf{R}|^2 = 1 - 1/N$, to first order,

$$\Lambda \partial_\Lambda \Delta = 1 - \frac{1}{K}\left(1 - \frac{1}{N}\right)\Delta. \qquad (21)$$

Hence, the phase transition at strong coupling terminates at $K_c = 1 - (1/N)$. The next order in RG for $N = 3$ is $\sim\Delta^2$ [similarly to Eq. (14) in the dual coupling g_2], while for $N \geq 4$ there are no Δ^2 terms since $\mathbf{R} \pm \mathbf{R}'$ are all longer than $\mathbf{R}$ and are therefore irrelevant at the transition. The next order is then $O(\Delta^3)$; hence, the critical line at strong coupling is $\Delta_c \sim \sqrt{K_c - K}$ with an infinite slope at K_c, for $N \geq 4$. This is similar to the $N \to \infty$ case [28], where $K_c = 1$ and $g_c^2 \sim 1/(1-K)$. The various phase boundaries are illustrated in Fig. 1.

The $N = 3$ case is self-dual; i.e., we find a relation [39] between $\langle B^i_\omega B^j_{-\omega} \rangle_{K,g_2}$ and $\langle B^i_\omega B^j_{-\omega} \rangle_{K \to K/3, g_2 \to \Delta}$. In particular, at the self-dual point $K = 1/\sqrt{3}, g_2 = \Delta$ on the critical line $\langle B^i_\omega B^j_{-\omega} \rangle$ is exactly given by the average of its values for $g_2 = 0$ and for $\Delta = 0$ (Fig. 1).

We proceed now to identify the order parameter of our phase transition, i.e., the transconductance. The phenomenon of current in chain i induced by a voltage on chain j has been studied in the context of crossed nanotubes [29–34]. In our case transconductance is a spontaneous order parameter and not a mechanical junction as for the nanotubes. The usual experiment is a two-probe type that for a single clean TLL yields [42,43] a conductance (e^2/h) determined by the normal leads [44–46]. For our system of N TLLs in the decoupled phase obviously $G_{ij} = (e^2/h)\delta_{ij}$ while in the coupled phase the strong generated coupling $\cos[\phi_i(0,\tau) - \phi_j(0,\tau)]$ forces the currents $I_i = \dot{\phi}_i(0,\tau)$ to be equal, $I_i = I$ with total dissipation $NI^2(h/e^2)$. We propose then that, with normal leads on each TLL, the resistance measured by a voltage in one wire is the sum of all individual resistances, hence

$$G_{i,j} = \begin{cases} \frac{e^2}{h} \delta_{i,j} & g_2 = 0 \\ \frac{e^2}{h} \frac{1}{N} & \Delta = 0. \end{cases} \qquad (22)$$

This implies that the transconductance exhibits a jump at the phase transition between these two values (Fig. 1).

To substantiate this rationale, we consider first a "local conductance" for the response to a field applied on a length L of a pure TLL [37]. The response function away from the impurity involves [39] $e^{\pm|\omega_n|x/u}$, a constant in the dc limit. Hence, $L \to 0$ can be taken, yielding

$$G^{\rm local}_{ij}(\omega) = \frac{-e^2}{\pi^2 \hbar} i(\omega + i\delta) \langle \phi_i(\omega_n) \phi_j(-\omega_n) \rangle |_{i\omega_n \to \omega + i\delta}.$$

In terms of the fields $B^i_\omega, \tilde{X}_\omega$ this becomes

$$G^{\rm local}_{i,j}(\omega) = \frac{e^2}{2\pi h} \omega \left[\langle B^i_\omega B^j_{-\omega} \rangle - \frac{1}{M\omega^2} \right]. \qquad (23)$$

From the sum rule on the B^i_ω correlations we obtain the exact sum rule $\sum_i G^{\rm local}_{i,j} = (e^2/hK)$. Using our results for the correlations, we obtain the dc local conductance at the fixed points and at the self-dual point

$$G^{\rm local}_{i,j} = \begin{cases} \frac{e^2}{h} K \delta_{i,j} & g_2 = 0 \\ \frac{e^2}{h} \frac{K}{N} & \Delta = 0 \\ \frac{1}{2} \frac{e^2}{h} K \left[\delta_{i,j} + \frac{1}{N} \right] & \mathrm{self-dual}\ (N = 2, 3). \end{cases} \qquad (24)$$

Along the transition line the conductance varies continuously: the correction to $G^{\rm local}_{i,j}$ is proportional to $1 - N\delta_{i,j}$ with a positive prefactor $\sim(K - 1/2)^2$ near $g_2 = 0$ and a negative one $\sim -(2/3-K)^2$ for $N=3$ and $\sim -[1-(1/N)-K]$ for $N \geq 4$ near $\Delta = 0$ [[39] Eq. (49)]. The extension to the inhomogeneous case with normal leads is shown in [39]; following ideas of the $N=1$ case [44–46] results in replacing $K \to 1$, yielding Eq. (22).

We have also considered an $N=2$ case with two coupled LLs, one with normal leads and the other a homogenous periodic TLL. We find [39] the conductance matrix $G_{ij} = e^2/h$, which also follows from our rationale since the TLL loop by itself has vanishing resistance. A similar problem was considered in [47] (see comparison in [39]).

We consider next the realization of our model by a superconducting (or BEC) grain. The Josephson coupling to s wave pairs in each TLL involves [37] $1/2ge^{iX_\tau - i\sqrt{2}\theta_{\rho,i}(0,\tau)}$, where X_τ is now the superconducting phase of the grain and $\theta_{\rho,i}$ is the dual phase to $\phi_{\rho,i}$ in the charge sector of chain i. The action in terms of $\theta_{\rho,i}$ has the same form as in Eqs. (3) and (4) with $K \to 1/(2K_\rho)$ [37] and $1/2M$ being the charging energy E_c of the grain; hence the phase diagram is also given by Fig. 1 with the axis being $1/(2K_\rho)$. Thus, for $N=2$ the phase boundary is at $K_\rho = 1$ and the strong coupled phase appears even for weakly attractive coupling $K_\rho > 1$. We note that the data [22] on a single wall carbon nanotube, expected to have $N=2$, shows with superconducting leads a surprisingly high supercurrent. If one of the leads contains a grain with not too small charging energy then our strong coupling phase, implying a strong Josephson coupling, would account for the data. For $N>2$, possible for nanotube ropes [22], the phase boundary interpolates between $K_\rho = 1$ and $K_\rho = [N/2(N-1)]$, allowing for a relevant Josephson coupling even in a range of repulsive interactions. The transconductance of this case needs a separate derivation [39], yet the result is the same as Eq. (24) except $K \to 2K_\rho$ and a prefactor 2 for this spinfull case.

Finally, from the decomposition (7) and the sum rule for the B^i_ω correlations within the effective model Eq. (13), we obtain the fluctuations of X_τ as $\langle |X_\omega|^2 \rangle = 1/M\omega^2$; i.e., they are not affected by the phase transition. Therefore, $\langle (X_\tau - X_{\tau'})^2 \rangle \sim |\tau - \tau'|$ justifies our assumption in deriving the action (6), i.e., that $|X_\tau - X_{\tau'}| \ll |\tau - \tau'|$.

We note that a finite impurity mass is essential for the derivation of our effective action (8), although it does not appear explicitly in the phase diagram. As seen from Eqs. (11) and (12), $1/M$ provides an upper limit on frequencies which implies an upper bound on temperature in a possible experiment, $T^* \approx (1/M) = (2\pi\rho_0)^2/M_0$. For Cs atoms [11] and TLL density $\pi\rho_0 = 4.5\ \mu\mathrm{m}^{-1}$ we find $T^* = 10^{-7}$ K; increasing the TLL density or reducing M_0 increase the range of $T < T^*$. For the realization with a superconducting grain $T^* \approx E_c$ where $E_c \approx 1$–10 K [26] is achievable in such devices. For $M \to \infty$ the problem reduces to a static impurity [40] with a phase transition at $K=1$ that separates a conducting phase from an insulating phase and no transconductance.

To realize a cold-atom experiment, one could consider an optical trap array of parallel tubes for the TLLs (as in Ref. [[11]]) and impurity atoms residing at the centers of the array's unit cells. The latter is possible by choosing the trapping frequency to be simultaneously red detuned for the TLL atoms and blue detuned for the impurity atoms, or vise versa, producing opposite sign couplings to the laser intensity; this arrangement was actually utilized [48] for Rb-Cs mixtures. To produce a reasonable impurity-TLL coupling we propose two routes. First, use atoms with a dipole moment (e.g., as in Er or Dy [49]). The long range dipole-dipole interaction provides the impurity-TLL interaction. A second route, with short range interactions, is to produce a cage type trap for the impurities that is shallow inside and allows a reasonable overlap with the TLL atoms, yet has barriers to keep the impurity in a given unit cell. The impurity-TLL interaction could then be enhanced by a Feshbach resonance as for the K-Rb case [9].

In conclusion, we have found that a dynamic boundary, such as a mobile impurity or a superconducting (or BEC) grain, coupled to N identical Luttinger liquids induces a phase transition for all $N \geq 2$. The order parameter is the conductance matrix, in particular a large transconductance appears in the strong coupling phase. In the superconducting grain case the strong coupling phase is also identified by a strong Josephson coupling, relevant to a number of active experimental setups [22,26,27].

B. H. thanks Y. Meir, E. Sela, I. Lerner, R. Folman, and G. Zaránd for stimulating discussions and Netanel Mirilashvili for his partaking in the 1/N expansion. B. H. also gratefully acknowledges funding by the German DFG through the DIP programme [FO703/2-1]. P. L. D. acknowledges support from ANR Grant No. ANR-17-CE30-0027-01 RaMaTraF. This work was supported in part by the Swiss National Science Foundation under Division II.

[1] A. Schirotzek, C.-H. Wu, A. Sommer, and M. W. Zwierlein, Phys. Rev. Lett. **102**, 230402 (2009).
[2] S. Nascimbéne, N. Navon, K. J. Jiang, L. Tarruell, M. Teichmann, J. McKeever, F. Chevy, and C. Salomon, Phys. Rev. Lett. **103**, 170402 (2009).
[3] B. Gadway, D. Pertot, R. Reimann, and D. Schneble, Phys. Rev. Lett. **105**, 045303 (2010).
[4] N. Spethmann, F. Kindermann, S. John, C. Weber, D. Meschede, and A. Widera, Phys. Rev. Lett. **109**, 235301 (2012).
[5] C. Zipkes, S. Palzer, C. Sias, and M. Kohl, Nature (London) **464**, 388 (2010).
[6] S. Schmid, A. Härter, and J. H. Denschlag, Phys. Rev. Lett. **105**, 133202 (2010).
[7] A. P. Chikkatur, A. Gorlitz, D. M. Stamper-Kurn, S. Inouye, S. Gupta, and W. Ketterle, Phys. Rev. Lett. **85**, 483 (2000).

[8] S. Palzer, C. Zipkes, C. Sias, and M. Kohl, Phys. Rev. Lett. **103**, 150601 (2009).
[9] J. Catani, G. Lamporesi, D. Naik, M. Gring, M. Inguscio, F. Minardi, A. Kantian, and T. Giamarchi, Phys. Rev. A **85**, 023623 (2012).
[10] T. Fukuhara, A. Kantian, M. Endres, M. Cheneau, P. Schauß, S. Hild, D. Bellem, U. Schollwöck, T. Giamarchi, C. Gross, I. Bloch, and S. Kuhr, Nat. Phys. **9**, 235 (2013).
[11] F. Meinert, M. Knap, E. Kirilov, K. Jag-Lauber, M. B. Zvonarev, E. Demler, and H.-C. Nägerl, Science **356**, 945 (2017).
[12] M. B. Zvonarev, V. V. Cheianov, and T. Giamarchi, Phys. Rev. Lett. **99**, 240404 (2007).
[13] M. Schecter, D. Gangardt, and A. Kamenev, Ann. Phys. (N. Y.) **327**, 639 (2012).
[14] F. Massel, A. Kantian, A. J. Daley, T. Giamarchi, and P. Törmä, New J. Phys. **15**, 045018 (2013).
[15] E. Burovski, V. Cheianov, O. Gamayun, and O. Lychkovskiy, Phys. Rev. A **89**, 041601 (2014).
[16] O. Gamayun, O. Lychkovskiy, and V. Cheianov, Phys. Rev. E **90**, 032132 (2014).
[17] A. Kantian, U. Schollwöck, and T. Giamarchi, Phys. Rev. Lett. **113**, 070601 (2014).
[18] F. Andraschko and J. Sirker, Phys. Rev. B **91**, 235132 (2015).
[19] E. V. H. Doggen, S. Peotta, P. Törmä, and J. J. Kinnunen, Phys. Rev. A **92**, 032705 (2015).
[20] A.-M. Visuri, T. Giamarchi, and P. Törmä, Phys. Rev. B **93**, 125110 (2016).
[21] D. Stornaiuolo, S. Gariglio, N. J. G. Couto, A. Fete, A. D. Caviglia, G. Seyfarth, D. Jaccard, A. F. Morpurgo, and J.-M. Triscone, Appl. Phys. Lett. **101**, 222601 (2012).
[22] A. Kasumov, M. Kociak, M. Ferrier, R. Deblock, S. Guéron, B. Reulet, I. Khodos, O. Stéphan, and H. Bouchiat, Phys. Rev. B **68**, 214521 (2003).
[23] A. Altland and R. Egger, Phys. Rev. Lett. **110**, 196401 (2013).
[24] K. Michaeli, L. A. Landau, E. Sela, and L. Fu, Phys. Rev. B **96**, 205403 (2017).
[25] L. Herviou, K. Le Hur, and C. Mora, Phys. Rev. B **94**, 235102 (2016).
[26] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygrd, P. Krogstrup, and C. M. Marcus, Nature (London) **531**, 206 (2016).
[27] S. Gazibegovic *et al.*, Nature (London) **548**, 434 (2017).
[28] B. Horovitz, T. Giamarchi, and P. Le Doussal, Phys. Rev. Lett. **111**, 115302 (2013).
[29] K. Flensberg, Phys. Rev. Lett. **81**, 184 (1998).
[30] Y. V. Nazarov and D. V. Averin, Phys. Rev. Lett. **81**, 653 (1998).
[31] R. Klesse and A. Stern, Phys. Rev. B **62**, 16912 (2000).
[32] V. V. Ponomarenko and D. V. Averin, Phys. Rev. Lett. **85**, 4928 (2000).
[33] A. Komnik and R. Egger, Eur. Phys. J. B **19**, 271 (2001).
[34] B. Gao, A. Komnik, R. Egger, D. C. Glattli, and A. Bachtold, Phys. Rev. Lett. **92**, 216804 (2004).
[35] H. Yi and C. L. Kane, Phys. Rev. B **57**, R5579 (1998).
[36] H. Yi, Phys. Rev. B **65**, 195101 (2002).
[37] T. Giamarchi, *Quantum Physics in One Dimension*, International Series of Monographs on Physics Vol. 121 (Oxford University Press, Oxford, 2004).
[38] For any correlation function, here we use the shorthand notation $\langle B^i_\omega B^j_{-\omega}\rangle$ to denote $\lim_{T\to 0}\{T\langle B^i_{\omega_n} B^j_{-\omega_n}\rangle\}$, where the complete $T=0$ correlation in continuum frequencies is $\langle B^i_\omega B^j_{\omega'}\rangle' = 2\pi\delta(\omega+\omega')\langle B^i_\omega B^j_{-\omega}\rangle$.
[39] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.166803 for derivations of the RG equations, correlation functions, duality and conductance which is either local or with normal leads in various geometries.
[40] C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. **68**, 1220 (1992); Phys. Rev. B **46**, 15233 (1992).
[41] V. Fateev, S. Lukyanov, A. Zamolodchikov, and A. Zamolodchikov, Phys. Lett. B **406**, 83 (1997).
[42] S. Tarucha, T. Honda, and T. Saku, Solid State Commun. **94**, 413 (1995).
[43] A. Yacoby, H. L. Stormer, N. S. Wingreen, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, Phys. Rev. Lett. **77**, 4612 (1996).
[44] I. Safi and H. J. Schulz, Phys. Rev. B **52**, R17040 (1995).
[45] D. L. Maslov and M. Stone, Phys. Rev. B **52**, R5539 (1995).
[46] V. V. Ponomarenko, Phys. Rev. B **52**, R8666 (1995).
[47] Y.-Z. Chou, A. Levchenko, and M. S. Foster, Phys. Rev. Lett. **115**, 186404 (2015).
[48] A. D. Lercher, T. Takekoshi, M. Debatin, B. Schuster, R. Rameshan, F. Ferlaino, R. Grimm, and H.-C. Nägerl, Eur. Phys. J. D **65**, 3 (2011).
[49] P. Ilzhöfer, G. Durastante, A. Patscheider, A. Trautmann, M. J. Mark, and F. Ferlaino, Phys. Rev. A **97**, 023633 (2018).