Rings and Coulomb boxes in dissipative environments

Yoav Etzioni,[1] Baruch Horovitz,[1] and Pierre Le Doussal[2]
[1]*Department of Physics, Ben Gurion University, Beer Sheva 84105 Israel*
[2]*CNRS-Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Cedex 05, Paris France*
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We study a particle on a ring in the presence of a dissipative Caldeira-Leggett environment and derive its response to a dc field. We show how this non-equilibrium response is related to a flux averaged equilibrium response. We find, through a two-loop renormalization group analysis, that a large dissipation parameter η flows to a fixed point $\eta^R = \hbar/(2\pi)$. We also reexamine the mapping of this problem to that of the Coulomb box and show that the relaxation resistance, of recent interest, is quantized for large η. For finite $\eta > \eta^R$ we find that a certain average of the relaxation resistance is quantized. We propose a Coulomb-box experiment to measure a quantized noise.

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I. INTRODUCTION

Two of the most important mesoscopic structures are rings, for the study of persistent currents, and quantum dots or boxes, for the study of charge quantization. Of particular recent interest is the quantization of the relaxation resistance, defined via an ac capacitance of a single-electron box (SEB). A SEB is defined as a quantum dot that has N_c transmission channels into a single-electron reservoir (i.e., an electrode) and is capacitively coupled to a gate voltage. This setup is equivalent to an RC circuit[1,2] whose capacitance at low frequency ω has the form $C_0(1 + i\omega C_0 R_q)$, identifying the relaxation resistance R_q. Following the prediction of Büttiker, Thomas, and Prêtre[1] that $R_q = h/(2e^2)$ for a single channel, a quantum mesoscopic RC circuit has been implemented in a two-dimensional electron gas[2] and $R_q = h/(2e^2)$ has been measured. The theory has been recently extended to include Coulomb blockade effects,[3,4] showing that $R_q = h/(2e^2)$ is valid for small dots and crosses over to $R_q = h/e^2$ for large dots.

In parallel, recent data has shown Aharonov-Bohm oscillations from single electron states in semiconducting rings.[5] Further theoretical works have considered the effects of dissipative environments on a single particle in a ring,[6] in particular studying the renormalization of the mass M^* and its possible relation to dephasing.[6–9] A related case of a ring coupled by tunneling to an electron lead has also been studied.[10]

It is rather remarkable that the ring and box problems are related via the Ambegaokar, Eckern, and Schön (AES) mapping[11] where the ring experiences a Caldeira-Leggett (CL)[12] environment. While the exact mapping assumes weak tunneling into the box with many channels, it has been extensively used to describe various tunnel junctions,[13] the Coulomb blockade phenomena in SEBs, and in the single electron transistor (SET).[13–22]

The ring problem is defined by a particle confined to a ring, coupled to a dissipative environment of the Caldeira-Leggett type, and in the presence of a field E, generated by a time dependent flux ϕ_x through the ring. This scenario is schematically illustrated in Fig. 1. The Caldeira-Leggett coupling can be realized, for example, by a normal metal whose mean-free path is much larger than the ring's radius.[9]

In the present work we address the ring problem by the real time Keldysh method and study it using a two-loop expansion and renormalization group (RG) reasoning. We find that perturbation theory identifies an unexpected small parameter $\sin[\hbar/(2\eta)]$, where η is the dissipation parameter on the ring, or the lead-dot coupling in the SEB. We infer that a large η flows to a fixed point η^R with $\hbar/(2\eta^R) = \pi$. While the thermodynamics of the ring-type problem has been much studied, including extensive Monte Carlo studies[17,20] of M^*, no sign of a finite-coupling fixed point has been detected. Our method evaluates the response to a strictly dc electric field E, equivalent to a magnetic flux through the ring that increases linearly with time; hence a nonequilibrium response. We claim that thermodynamic quantities like M^*, that are flux sensitive decouple from the response to E, a response that averages over flux values. This general relation between nonequilibrium and equilibrium responses is given by Eq. (39) below. This relation has been noticed for a model with particle tunneling between a ring and an environment.[23]

In terms of the SEB, our results extend the previous analysis[3,4] to the case of many channels N_c, an experimentally realizable scenario.[24] We note that for $N_c > 1$ the relaxation resistance for noninteracting electrons[1] becomes $h/(2N_c e^2)$. We find that for strong coupling, $\eta/\hbar \gtrsim 1$, the relaxation resistance is quantized to e^2/h up to an exponentially small correction $\sim e^{-\pi\eta/\hbar}$. For finite η, but still $\eta > \eta^R$ we find that a certain average of the relaxation resistance is quantized [see Eq. (82)].

The present work considerably expands on our previous letter.[25] In Sec. II we present the ring and box models, with some exact general properties. In Sec. III we present RG and numerical solutions for the semiclassical case, while Sec. IV presents the perturbation and RG analysis of the full quantum case. The discussion in Sec. V summarizes our results, discusses its topological interpretation, and details a proposed Coulomb-box experiment to detect our predicted quantized noise. The appendices give details of the ring-box mapping and of the various perturbation expansions. We consider temperature $T = 0$ throughout.

As a simple motivation for our main result, we present here a topological interpretation of the fixed point η^R, based on the Thouless charge pump concept.[26] Consider a slow

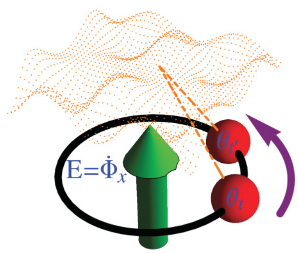

FIG. 1. (Color online) Artist's view of a particle on a ring, coupled to an environment, with a field $E = \dot{\phi}_x$ due to a time dependent flux through the ring. The particle polarizes the environment which in turn modifies the motion of the particle at later times (i.e., an effective nonlocal interaction).

change of ϕ_x by one unit with $\hbar\dot{\phi}_x = \eta^R\langle\dot{\theta}\rangle$. For the special value $\eta^R = \hbar/(2\pi)$ the total change in the position of the particle is $\int_t \langle\dot{\theta}\rangle dt = 2\pi$; that is, the particle comes back to the same position on the ring and a unit charge is transported.

II. MODEL AND GENERAL PROPERTIES

A. Semiclassical model

We derive first a Langevin equation for a particle on a ring. Consider the standard Langevin equation for a particle with coordinate x_t in one dimension of the form

$$R^{-1}_{t,t'}x_{t'} = \xi_t, \tag{1}$$

where ξ_t is a Gaussian random force from an environment, where the average on the environment degrees of freedom is

$$\langle\xi_t\xi_{t'}\rangle = B_{t,t'}. \tag{2}$$

This relation defines a linear response for either $x_\omega = R_\omega\xi_\omega$ or $\xi_\omega = R^{-1}_\omega x_\omega$, after Fourier transforms [e.g., R_ω is the Fourier transform of $R_t = R_{t,0}$]. Hence the fluctuation dissipation theorem (FDT) at temperature T can be applied either way, leading to

$$\begin{aligned} K_x(\omega) &= \hbar\coth\left(\tfrac{1}{2}\beta\hbar\omega\right)\mathrm{Im}[R_\omega],\\ B_\omega &= \hbar\coth\left(\tfrac{1}{2}\beta\hbar\omega\right)\mathrm{Im}\frac{-1}{R_\omega}, \end{aligned} \tag{3}$$

where $K_x(\omega)$ is the Fourier transform of $K_x(\tau) = \frac{1}{2}\langle x_t x_{t+\tau} + x_{t+\tau}x_t\rangle$. The simplest choice corresponds to a particle with mass m and friction coefficient η, so that at temperature $T = 0$,

$$\begin{aligned} &m\ddot{x}_t + \eta\dot{x}_t = \xi_t,\\ &R_0(\omega) = \frac{-1}{m\omega^2 + i\omega\eta}, \quad R_0(t) = \frac{1}{\eta}[1 - e^{-\eta t/m}]\Theta(t),\\ &B_\omega = \hbar\eta|\omega|, \quad B_t = \frac{-\hbar\eta}{\pi t^2} \quad (t \neq 0), \end{aligned} \tag{4}$$

where $\Theta(t)$ is the Heaviside function and $R_0(t-t')$ is the response in this case. While the mass provides a high-frequency cutoff which we denote $\omega_c = \eta/m$, the singularity of B_t at $t = 0$ implies the need for an additional cutoff. This additional cutoff is a convenience and will be used below in the simulations as well as in the RG derivation. A method for deriving general response functions is based on Kramers-Kronig relations.[27] In the notation of Eq. (2.7) of Ref. 27 we choose $\mathrm{Re}\mu(\omega) = \eta/(1+\omega^2\tau_0^2)$ so that the response function $R^{-1}_{t-t'}$, after Fourier transform, is

$$R^{-1}_\omega = -m\omega^2 - \frac{i\omega\eta}{1 - i\omega\tau_0}. \tag{5}$$

To justify the use of this form it suffices to say that it has the remarkable and necessary property that both R_ω and R^{-1}_ω have no poles in the upper half plane, as needed for causal functions; [note that R_ω reduces to $R_0(\omega)$ when $\tau_0 = 0$]. The FDT at $T = 0$ gives

$$B_\omega = \frac{\hbar|\omega|\eta}{1 + \omega^2\tau_0^2}, \tag{6}$$

so that $1/\tau_0$ provides a cutoff for the environment frequencies, in addition to the cutoff $m/\eta = 1/\omega_c$. Hence for $4\tau_0 < m/\eta$ ($\delta \to +0$),

$$\begin{aligned} &R_t = \Theta(t)\frac{\tau_0}{m}\left[\frac{m}{\eta\tau_0}e^{-\delta t} + \frac{1-\lambda_1}{\lambda_1 x}e^{-\lambda_1 t/\tau_0} - \frac{1-\lambda_2}{\lambda_2 x}e^{-\lambda_2 t/\tau_0}\right],\\ &\lambda_1 = \frac{1}{2}[1+x], \quad \lambda_2 = \frac{1}{2}[1-x], \quad x = \sqrt{1 - \frac{4\eta\tau_0}{m}}, \end{aligned} \tag{7}$$

while for $4\tau_0 > m/\eta$ with $x = \sqrt{4\eta\tau_0/m - 1}$,

$$\begin{aligned} R_t = \Theta(t)\frac{1}{\eta}\bigg\{e^{-\delta t} - \bigg[&\frac{1-x^2}{2x}\sin(xt/2\tau_0)\\ &+ \cos(xt/2\tau_0)\bigg]e^{-t/2\tau_0}\bigg\}. \end{aligned} \tag{8}$$

Consider now the two-dimensional system and its projection on a ring [i.e., $\mathbf{x}_t = (\cos\theta_t, \sin\theta_t)$], so that θ_t is the angular position of the particle and the radius is chosen as unity. In cartesian coordinates we define random forces in the x, y directions so that $R^{-1}_{t-t'}\cos\theta_{t'} = -\xi^b_t$, $R^{-1}_{t-t'}\sin\theta_{t'} = \xi^a_t$. The ring potential confines the motion to the azimuthal part, so that only the tangent force $-\xi^a_t\cos\theta_t + \xi^b_t\sin\theta_t$ is allowed, hence

$$\begin{aligned} &-\sin\theta(t)R^{-1}_{t-t'}\cos\theta_{t'} + \cos\theta_t R^{-1}_{t-t'}\sin\theta_{t'}\\ &\quad = \xi^a_t\cos\theta_t + \xi^b_t\sin\theta_t + E, \end{aligned} \tag{9}$$

where ξ^a_t, ξ^b_t are independent and each have the correlations of Eq. (2). An external tangent electric field E has been added corresponding to a flux through the ring that is increasing linearly with time $\phi_x = Et$. With $R_0(t-t')$ given by Eq. (4) the differential form $R^{-1}_0(t) = mr\partial_t^2 + \eta r\partial_t$, can be used leading to

$$m\ddot{\theta}_t + \eta\dot{\theta}_t = \xi^a_t\cos\theta_t + \xi^b_t\sin\theta_t + E. \tag{10}$$

This nonlinear Langevin equation has been studied also in the SET context.[28] Comparing the time derivatives in Eq. (10) identifies a cutoff frequency $\omega_c = \eta/m$. At $\omega > \omega_c$ the mass term dominates while at $\omega < \omega_c$ the environment dominates, leading to renormalizations. The nonlinear Langevin equation (10), including an average on the random forces, is equivalent to a partition function

$$Z = \int \mathcal{D}[\theta,\xi]\,\delta\big(m\ddot{\theta}_t + \eta\dot{\theta}_t - \xi_t^a\cos\theta_t - \xi_t^b\sin\theta_t - E\big)\\ \times \exp\left\{-\int_\omega \big[|\xi_\omega^a|^2 + |\xi_\omega^b|^2\big]/(2B_\omega)\right\}. \tag{11}$$

Introducing the "quantum" field $\hat{\theta}$ by $\delta(X_t) = \int \mathcal{D}[\hat{\theta}]e^{i\hat{\theta}_t X_t}$, and averaging over the noise field ξ_x, ξ_y results in the semiclassical partition function $Z = \int \mathcal{D}[\theta,\hat{\theta}]e^{-S[\theta,\hat{\theta}]}$ where $S[\theta,\hat{\theta}] = S_0 + S_{\rm int}$ is given by the t, t' integrations

$$\begin{aligned} S_0 &= i\int_{t,t'} \hat{\theta}_t (R_{t,t'})^{-1}\theta_{t'} - iE\int_{t'}\hat{\theta}_{t'}\\ &= i\int_\omega R_\omega^{-1}\hat{\theta}_\omega\theta_{-\omega} - iE\int_{t'}\hat{\theta}_{t'}, \\ S_{\rm int} &= \frac{1}{2}\int_{t,t'}\hat{\theta}_t B_{t,t'}\hat{\theta}_t\cos(\theta_t - \theta_{t'}). \end{aligned} \tag{12}$$

This has the form of a Keldysh action, with $\theta, \hat{\theta}$ being the classical and quantum fields, respectively. We will see below that this action is the semiclassical $\hbar \to 0$ limit of the full quantum system.

B. Quantum model

We proceed to define the full quantum problem. The one-dimensional Langevin system[12,29,30] has the Keldysh partition $Z = \int \mathcal{D}\hat{x}_t\mathcal{D}x_t e^{-S_K}$ where

$$S_K = i\int_{t,t'}\hat{x}_t R_{t,t'}^{-1}x_{t'} + \frac{1}{2}\int_{t,t'}\hat{x}_t B_{t,t'}\hat{x}_{t'} \tag{13}$$

and $\hat{x}_t, x_t$ are the quantum and classical fields, respectively,

$$x_t = \frac{1}{2}(x_t^+ + x_t^-), \quad \hat{x}_t = \frac{1}{\hbar}(x_t^+ - x_t^-), \tag{14}$$

and $x_t^\pm$ are on the upper and lower Keldysh contour, respectively. On a ring, we use a two-dimensional vector notation

$$\mathbf{x}_t^+ = [\cos\theta_t^+, \sin\theta_t^+], \quad \mathbf{x}_t^- = [\cos\theta_t^-, \sin\theta_t^-]. \tag{15}$$

Defining

$$\theta_t = \frac{1}{2}(\theta_t^+ + \theta_t^-), \quad \hat{\theta}_t = \frac{1}{\hbar}(\theta_t^+ - \theta_t^-), \tag{16}$$

and using trigonometric identities we obtain the quantum action

$$S_K = i\frac{2}{\hbar}\int_{t,t'} R_{t,t'}^{-1}\sin\left(\frac{\hbar}{2}\hat{\theta}_t\right)\cos\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right)\sin(\theta_{t'} - \theta_t)\\ + \frac{2}{\hbar^2}\int_{t,t'} B_{t,t'}\sin\left(\frac{\hbar}{2}\hat{\theta}_t\right)\sin\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right)\cos(\theta_{t'} - \theta_t). \tag{17}$$

We note that the path integral involves continuous θ_t trajectories that can involve n rotations around the ring. Consider the time evolution from an initial wave function $\psi(\theta_0,t_0)$ at time t_0 to a final state $\psi(\tilde{\theta}_t,t)$, where both initial and final angles are compact, $0 < \theta_0,\tilde{\theta}_t < 2\pi$,

$$\psi(\tilde{\theta}_t,t) = \int_0^{2\pi} d\theta_0 \sum_n \int_{\theta_0}^{\tilde{\theta}_t + 2\pi n}\mathcal{D}\theta e^{-S(t,t_0)}\psi(\theta_0,t_0). \tag{18}$$

The sum on the integers n expresses that the probability to arrive at a given $\tilde{\theta}_t$ is a sum of probabilities, each with n rotations. The path integral can therefore be written in terms of a decompactified variable $\theta_t = \tilde{\theta}_t + 2\pi n$ (i.e., $\sum_n \int_{\theta_0}^{\tilde{\theta}_t+2\pi n}\mathcal{D}\theta \to \int_{\theta_0}^{\theta_t}\mathcal{D}\theta$ where now $-\infty < \theta_t < \infty$). This shift does not affect the periodic forms in Eq. (17); however, it does affect an external electric field E. Consider a time-dependent flux $\phi_x(t) = Et$ that contributes to the action a term

$$\int_{t_i}^{t_f}\phi_x(t)\dot{\theta}_t dt = -E\int_{t_i}^{t_f}\theta_t dt + \phi_x(t_i)\theta_{t_i} - \phi_x(t_f)\theta_{t_f}.$$

The partial integration is allowed only for the decompactified variable θ_t (i.e., the work done by E is finite for each 2π rotation). The boundary terms are neglected; for example, one can choose $\phi_x(t_i) = \phi_x(t_f) = 0$ where $t_i, t_f \to -\infty$ are boundary times on a Keldysh contour; the field E is turned on slowly away from these times.

In the following we will consider a perturbative scheme with a field E and a bare velocity $v = E/\eta$ and with θ_t decomposed to $\theta_t = \delta\theta_t + vt$ [the true velocity is defined below as $v^R(E) = \langle\dot{\theta}_t\rangle$]. The velocity v provides a low-frequency cutoff eliminating divergence of the perturbative expansion and eventually allows for RG treatment. It will be convenient to use the two-cutoff response (5) with $R_\omega^{-1} = -m\omega^2 + \delta R_\omega^{-1}$, where $\delta R_\omega^{-1} = -i\omega\eta/(1 - i\omega\tau_0)$, hence

$$\begin{aligned} \delta R_{t,t'}^{-1} &= \partial_{t'}\int_\omega \frac{-\eta}{1 - i\omega\tau_0}e^{-i\omega(t-t')}\\ &= -\frac{\eta}{\tau_0}\partial_{t'}[e^{-(t-t')/\tau_0}\Theta(t - t')]\\ &= \frac{\eta}{\tau_0}e^{-(t-t')/\tau_0}\Theta(t - t')\partial_{t'}. \end{aligned} \tag{19}$$

The operator identity is satisfied for any function decaying faster then $e^{|t'|/\tau_0}$ at $t' \to -\infty$. Note,

$$i\int_{t,t'}\hat{\theta}_t\delta R_{t,t'}^{-1}vt' = i\frac{\eta v}{\tau_0}\int_t\hat{\theta}_t\int_{-\infty}^t e^{-(t-t')/\tau_0}dt' = iv\eta\int_t\hat{\theta}_t. \tag{20}$$

The mass term with $m\omega^2 \to \delta(t - t')\partial_t\partial_{t'}$ produces $m\int_t\dot{\hat{\theta}}_t\dot{\theta}_t = m\int_t\dot{\hat{\theta}}_t\delta\dot{\theta}_t + mv\int_t\dot{\hat{\theta}}_t$; the last term with $mv = E/\omega_c$ is neglected relative to the field term $\int_t Et\dot{\hat{\theta}}_t$. The full action is then

$$\begin{aligned} S_K &= S_0 + S_{\rm int} + S_c,\\ S_0 &= i\int_{t,t'}\hat{\theta}_t R_{tt'}^{-1}\theta_{t'} - iE\int_t\hat{\theta}_t = i\int_{t,t'}\hat{\theta}_t R_{tt'}^{-1}\delta\theta_{t'},\\ S_{\rm int} &= \frac{2}{\hbar^2}\int_{t,t'}B_{t,t'}\sin\left(\frac{\hbar}{2}\hat{\theta}_t\right)\sin\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right)\cos(\theta_{t'} - \theta_t),\\ S_c &= i\frac{2}{\hbar}\int_{t,t'}\delta R_{t,t'}^{-1}\\ &\quad\times\left[\sin\left(\frac{\hbar}{2}\hat{\theta}_t\right)\cos\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right)\sin(\theta_{t'} - \theta_t) - \frac{\hbar}{2}\hat{\theta}_t\theta_{t'}\right]. \end{aligned} \tag{21}$$

The use of a single cutoff (4) with

$$R_0^{-1}(t,t') = \delta(t-t')[m\partial_t\partial_{t'} + \eta\partial_{t'}] \tag{22}$$

leads to a simpler action. It corresponds to $\tau_0 \to 0$, hence $\delta R_{t,t'}^{-1} \to \eta\delta(t-t')\partial_{t'}$,

$$\frac{2}{\hbar}R_0^{-1}(t,t')\sin\left(\frac{\hbar}{2}\hat{\theta}_t\right)\cos\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right)\sin(\theta_{t'}-\theta_t)$$
$$= \delta(t-t')\left[m\dot{\hat{\theta}}_t\dot{\theta}_t + \frac{\eta}{\hbar}\sin(\hbar\hat{\theta}_t)\dot{\theta}_{t^-}\right], \tag{23}$$

where t^- is infinitesimally below t so that the retarded nature of $R_{t,t'}^{-1}$ is maintained. The action $S_K = S_0 + S_{\text{int}} + S_c$ is then

$$S_0 = i\int_{t,t'}\hat{\theta}_t R_0^{-1}(t,t')\delta\theta_{t'} = i\int_t [m\dot{\hat{\theta}}_t\delta\dot{\theta}_t + \eta\hat{\theta}_t\delta\dot{\theta}_t]$$
$$= i\int_t [m\dot{\hat{\theta}}_t\dot{\theta}_t + \eta\hat{\theta}_t\dot{\theta}_t] - iE\int_t\hat{\theta}_t,$$
$$S_{\text{int}} = \frac{2}{\hbar^2}\int_{t,t'}B_{t,t'}\sin\left(\frac{\hbar}{2}\hat{\theta}_t\right)\sin\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right)\cos(\theta_{t'}-\theta_t),$$
$$S_c = \frac{i\eta}{\hbar}\int_t[\sin(\hbar\hat{\theta}_t)\dot{\theta}_{t^-} - \hbar\hat{\theta}_t\dot{\theta}_{t^-}], \quad \tau_0 \to 0. \tag{24}$$

Note that this action reduces to that of the semiclassical case (12) when $\hbar \to 0$.

C. Renormalized friction

The renormalized friction $\eta^R(E)$ is defined by the renormalized response $R_{t,t'}^R = i\langle\theta_t\hat{\theta}_{t'}\rangle_E$ and its dc limit:

$$\frac{1}{\eta^R(E)} = \lim_{\omega\to 0}\left(-i\omega R_\omega^R\right), \tag{25}$$

in analogy with the bare form (4). We show now that the renormalized $\eta^R(E)$ is also the local slope of dv^R/dE, where v^R is the E-dependent renormalized velocity

$$v^R \equiv \langle\dot{\theta}_t\rangle = \int\mathcal{D}[\theta]\dot{\theta}_t e^{-S_K}. \tag{26}$$

Therefore,

$$\frac{dv^R}{dE} = i\left\langle\int_{t'}\dot{\theta}_t\hat{\theta}_{t'}\right\rangle = \int_{t'}\frac{d}{dt}R_{t,t'}^R = \int_{t'}\int_\omega(-i\omega)R_\omega^R e^{-i\omega(t-t')}$$
$$= \lim_{\omega\to 0}\frac{-i\omega}{-i\eta^R(E)\omega} = \frac{1}{\eta^R(E)}. \tag{27}$$

In particular we are interested in the limit $\eta^R = \eta^R(E\to 0)$.

We show now an alternative procedure for evaluating η^R. Consider the Keldysh partition $Z = \int\mathcal{D}[\theta]e^{-S_K}$ and shift $\hat{\theta}_t \to \hat{\theta}_t + a_t$. The result must be a_t independent, and by choosing the form (23) with $\tau_0 \to 0$ (the following identity is actually independent of cutoff choices),

$$0 = \left.\frac{\delta Z}{\delta a_t}\right|_0 = -\left\langle\frac{\delta(S_0+S_{\text{int}}+S_c)}{\delta\hat{\theta}_t}\right\rangle = -i(\eta v^R - E - \delta E),$$
$$\delta E \equiv i\left\langle\frac{\delta(S_{\text{int}}+S_c)}{\delta\hat{\theta}_t}\right\rangle, \tag{28}$$

since $-i\langle\delta S_0/\delta\hat{\theta}_t\rangle = -m\langle\ddot{\theta}_t\rangle + \eta\langle\dot{\theta}\rangle - E$ and v^R is time independent, at least for long times.

Taking an E derivative of Eq. (28) and using Eq. (27) we obtain

$$\frac{1}{\eta^R(E)} = \frac{1}{\eta} + \frac{1}{\eta^2}\frac{\partial}{\partial v}\delta E. \tag{29}$$

We have checked, up to second-order terms, that the results of Eqs. (27) and (29) coincide. The use of Eq. (29) is technically easier.

D. Equilibrium correlations

In this section we consider the equilibrium response to a change in flux and derive a relation with the nonequilibrium response to a field.

Consider now the form of $\tilde{K}(\omega)$ as a response to a flux ϕ_x. Linear response to $\delta\mathcal{H}_{\text{ring}} = +\hbar\dot{\theta}\delta\phi_x(t)$ is

$$\hbar\langle\dot{\theta}\rangle = -\int_{t'}\tilde{K}_{t,t'}\delta\phi_x(t'). \tag{30}$$

This corresponds also to the velocity correlation

$$\tilde{K}_{t,t'} = +i\theta(t-t')\langle[\dot{\theta}_t,\dot{\theta}_{t'}]\rangle. \tag{31}$$

We expect that the dc response is positive for small ϕ_x, so we define

$$\tilde{K}(\omega) = -K_0(\phi_x) + i\omega K_1(\phi_x) + O(\omega^2). \tag{32}$$

The response $K_0(\phi_x)$ is the persistent current; that is, for a static flux one can integrate Eq. (30):

$$\langle\dot{\theta}\rangle = \int_0^{\phi_x}K_0(\phi_x')d\phi_x'. \tag{33}$$

The periodicity of the persistent current implies $\int_0^1 K_0(\phi_x)d\phi_x = 0$. The curvature of the free energy F (or energy at $T=0$) at $\phi_x = 0$ is a well-studied object.[6–9] For general ϕ_x it is defined by a Matsubara imaginary time connected correlation

$$\frac{1}{\hbar}\frac{\partial^2 F}{\partial\phi_x^2} = (\beta)^{-1}\int_0^\beta\int_0^\beta\langle\dot{\theta}_\tau\dot{\theta}_{\tau'}\rangle^c d\tau d\tau' = K_0(\phi_x), \tag{34}$$

where $K(i\omega_n = 0) = +K_0$ (there is a sign difference in the standard Matsubara notation). An effective mass is defined by $K_0(0) = \hbar/M^*$ so that $M^* = m$ without interactions, while for strong $\eta \gg 1$ coupling $M^* \sim e^{\pi\eta}$ is exponentially large.[6–9]

To appreciate the role of K_1 consider FDT for the symmetrized correlation at small ω

$$\langle|\dot{\theta}_\omega|^2\rangle^{\text{sym}} = \text{sgn}(\omega)\text{Im}\tilde{K}_\omega = |\omega|K_1. \tag{35}$$

The diffusion involves the response $\langle|\theta_\omega|^2\rangle = K_1/|\omega|$, hence for $t\to\infty$

$$\langle(\theta_t-\theta_0)^2\rangle = K_1\int d\omega\frac{1-\cos\omega t}{\pi|\omega|} = \frac{2K_1}{\pi}\ln(\omega_x t), \tag{36}$$

where ω_x is a characteristic frequency where higher-order terms in ω terms set in.

Consider now the linear response to an electric field $\delta\mathcal{H}_{\rm ring} = -E(t)\theta_t$ and use the response $\langle\theta_t\rangle = R^R_{t,t'}E(t')$. The definition (25) implies that the low-ω limit has the form $R^R_\omega = -1/(i\omega\eta^R)$. Since $E = \hbar\dot{\phi}_x$ we expect $\hbar\omega^2 R^R_\omega = \tilde{K}(\omega)$. However, there is a difficulty with the latter relation, if taken literally,

$$\frac{-\hbar\omega^2}{i\omega\eta^R}?=?-K_0(\phi_x) + i\omega K_1(\phi_x). \tag{37}$$

It is also not clear which ϕ_x to use in this relation. To resolve this issue consider the $\tilde{K}$ response with a constant electric field

$$\hbar\langle\dot{\theta}_t\rangle = -\int_{t'}\tilde{K}_{t,t'}Et'. \tag{38}$$

Note first that an additional constant ϕ_x in $Et'/\hbar + \phi_x$ can be eliminated by redefining the origin of the time t', hence the persistent current part should be eliminated. More precisely, define $\phi_x(t) = Et/\hbar$; the $\omega = 0$ component $K_0(\phi_x) = K_0(Et/\hbar)$ becomes a periodic function [i.e., an ac response with frequency $\omega_E = (2\pi/\hbar)E$]. For $\omega \to 0$ this persistent current response averages to zero (i.e., $\int_0^1 K_0(\phi_x)d\phi_x = 0$). The same reasoning applies to a ϕ_x average on $K_1(\phi_x)$. Hence for the purpose of evaluating the dc response of Eq. (25) we need to average on the flux in Eq. (32), hence

$$\lim_{E\to 0}\lim_{\omega\to 0}\frac{\tilde{K}(\omega)}{i\omega} = \int_0^1 K_1(\phi_x)d\phi_x = \frac{\hbar}{\eta^R}. \tag{39}$$

The order of limits in Eq. (5) signifies that η^R is essentially a nonequilibrium response. The equilibrium-nonequilibrium relation (39) has been noticed in the solution of a Boltzmann relaxation equation for particles on a ring, allowing for particle tunneling into an environment.[23]

The physical picture is that in a dc field the particle rotates around the ring and produces two types of currents. First is the persistent current that oscillates in time as ϕ_x increases and is therefore time averaged to zero; this current is nondissipative. Second, there is a genuine dc response from the $i\omega K_1$ term, which is dissipative.

E. Coulomb box

Consider now the Coulomb-box system; namely, a finite region (a "dot") with charging energy E_c coupled by tunneling to a single metallic lead. The Hamiltonian is

$$\mathcal{H} = \sum_k \epsilon_k a^\dagger_{k,i}a_{k,i} + \sum_{\alpha,i}\epsilon_\alpha d^\dagger_{\alpha,i}d_{\alpha,i} + E_c(\hat{N} - N_0)^2 + \sum_{k,\alpha,i} t_{k,\alpha,i}a^\dagger_{k,i}d_{\alpha,i} + \text{H.c.}, \tag{40}$$

where $i = 1,\ldots,N_c$ are channel indices, $d_{\alpha,i}$ are dot electron operators with spectra ϵ_α, $a_{k,i}$ are lead electron operators with spectra ϵ_k, $\hat{N} = \sum_{\alpha,i} d^\dagger_{\alpha,i}d_{\alpha,i}$ is the number operator on the dot, $E_c = e^2/(2C_g)$ is the charging energy with C_g is the geometric (bare) capacitance, and N_0 is the gate voltage in units of $2E_c$. The channel index i is diagonal in the tunneling term (i.e., corresponds to transverse modes that are conserved in tunneling).

Consider the density correlations

$$K_{t,t'} = +i\theta(t-t')\langle[\hat{N}_t,\hat{N}_{t'}]\rangle. \tag{41}$$

The AES mapping to the ring problem is reproduced in Appendix A. In particular, N_0 corresponds to $-\phi_x$, $2E_c$ to $\hbar^2/m$, and the relation to the velocity correlation on the ring is

$$\hbar^2\tilde{K}_{t,t'} = -2E_c\hbar\delta(t-t') + 4E_c^2 K_{t,t'}. \tag{42}$$

Using the notation[3] $K(\omega) = \hbar C_0(1 + i\omega C_0 R_q)/e^2$, where C_0 is the renormalized capacitance and R_q is the relaxation resistance, we obtain

$$\hbar\tilde{K}(\omega) = -2E_c + 4E_c^2\frac{C_0}{e^2}(1 + i\omega C_0 R_q). \tag{43}$$

Hence the mapping between the Coulomb box and the ring for the curvature is, using Eq. (34),

$$\frac{\hbar^2}{M^*(\phi_x)} = \hbar K_0(\phi_x) = 2E_c\left(1 - \frac{C_0}{C_g}\right)$$
$$\Rightarrow \frac{m}{M^*(\phi_x)} = 1 - \frac{C_0(N_0)}{C_g}, \tag{44}$$

while for the dissipation, using Eq. (39),

$$\frac{\hbar}{\eta^R} = \int_0^1 K_1(\phi_x)d\phi_x = \frac{e^2}{\hbar}\int_0^1\frac{C_0^2(N_0)}{C_g^2}R_q(N_0)dN_0. \tag{45}$$

We note that $\int_0^1[C_0(N_0)/C_g]dN_0 = 1$ due to the periodicity of $F(\phi_x)$. An extensive study[6–9] of $M^*(0)$ shows that it satisfies $M^*(0) > m$ and that for large η (the bare interaction parameter) $M^*(0)/m \sim e^{\pi\eta} \gg 1$. Hence,

$$\frac{C_0}{C_g} = 1 - O(e^{-\pi\eta}), \quad \eta \gtrsim 1, \tag{46}$$

and $C_0 \to C_g$ for large η.

At this stage we can already propose an interesting experiment for the SEB. By analogy with $E = \hbar\dot{\phi}_x$ in the ring, we propose measuring the response to a gate voltage that is linear in time $N_0 \sim t$. This leads to a dc current into the Coulomb box whose dissipation is the average in Eq. (45). This average is predicted to be quantized, at least for $\eta > \eta^R$, as shown below.

III. SEMICLASSICAL RENORMALIZATION GROUP AND NUMERICS

A. Perturbations and renormalization group

We study here the action (12) with a perturbation series for correlation functions. Consider first the correlation $C_{t',t} = \langle\theta_{t'}\theta_t\rangle$, which to first order is

$$C^{(1)}_{t,t'} = \langle\theta_{t'}\theta_t(-S_{\rm int})\rangle_{S_0} = \int_{t_1,t_2} B_{t_1,t_2}\cos v(t_1 - t_2)R_{t,t_1}R_{t',t_2}. \tag{47}$$

In Fourier space

$$C^{(1)}_\omega = |R_\omega|^2 B^v_\omega, \tag{48}$$

where $B^v_\omega = \frac{1}{2}(B_{\omega+v} + B_{\omega-v})$. Since $C^{(1)}_{t'=t}$ is divergent it is useful to evaluate $\tilde{C}_{t,t'} = \langle[\theta_t - \theta_{t'}]^2\rangle$, which to first order is, with $\tau = t - t'$ ($\tau \gg 1/\omega_c$),

$$\tilde{C}_\tau = \int_\omega B^v_\omega |R_\omega|^2 (1 - \cos\omega\tau) \approx \frac{2\hbar}{\pi\eta} \begin{cases} \ln\left(\frac{\eta\tau}{m}\right), & \tau < 1/v \\ \frac{1}{2}\pi v\tau, & 1/v < \tau. \end{cases} \tag{49}$$

For $E = 0$ the angular position diffuses logarithmically, while for $E \neq 0$ the long-time fluctuation is linear in time.

Consider next the response function to second order in S_{int},

$$R^R_{t,t'} = i\langle\hat{\theta}_{t'}\theta_t\rangle = R_{t,t'} + R^{(1)}_{t,t'} + R^{(2)}_{t,t'} = R_{t,t'} + i\left\langle\hat{\theta}_{t'}\theta_t\left(-S_{\text{int}} + \frac{1}{2}S^2_{\text{int}}\right)\right\rangle_{S_0}. \tag{50}$$

Note that the disconnected terms in the perturbation $\langle S^n_{\text{int}}\rangle_{S_0}$ vanish for any order n, due to the normalization $Z = 1$. The first-order response function is

$$R^{(1)}_{t,t'} = -i\frac{1}{2}\int_{t_1,t_2} B_{t_1,t_2}\langle\hat{\theta}_{t_1}\hat{\theta}_{t_2}\cos\left(\theta_{t_1} - \theta_{t_2}\right)\hat{\theta}_{t'}\theta_t\rangle_{S_0}. \tag{51}$$

The result in the frequency variable is (see Appendix B)

$$R^{(1)}_\omega = R^2_\omega \int_{\omega_1} R_{\omega_1}\left[B^v_{\omega_1} - B^v_{\omega-\omega_1}\right] = R^2_\omega \int_t R_t B_t \cos vt(e^{i\omega t} - 1). \tag{52}$$

We note that for $v = 0$ FDT is maintained, to this order, $C^{(1)}_\omega|_{v=0} = \text{Im}R_\omega\hbar\,\text{sgn}(\omega)$.

The renormalized η to first order is then

$$\frac{1}{\eta^R_1} = \lim_{\omega\to 0}(-i\omega)R^{(1)}_\omega = \lim_{\omega\to 0}\frac{-i\omega}{(-i\omega)^2\eta^2}\int_t R_t B_t \cos vt(i\omega t) = \frac{1}{2\eta^2}\ln\left(1 + \omega_c^2/v^2\right) = -\frac{\ln v/\omega_c}{\eta^2} + O(v). \tag{53}$$

Considering next the second order in Eq. (50) we obtain (see Appendix B)

$$\begin{aligned} R^{(2)}_\omega = R^2_\omega\bigg(&-\frac{1}{2}\int_t R_t B_t \cos vt(e^{i\omega t} - 1)\tilde{C}^{(1)}_t \\ &+ \int_t R^{(1)}_t B_t \cos vt(e^{i\omega t} - 1) \\ &+ R_\omega\left[\int_t R_t B_t \cos vt(e^{i\omega t} - 1)\right]^2 \\ &- \int_{t_1,t_2} R_{t_1}B_{t_1}B_{t_2}\sin vt_1 \sin vt_2(1 - e^{i\omega t_1})t_1\bigg). \end{aligned} \tag{54}$$

Denoting the contribution of the last term in Eq. (54) as $\delta(1/\eta^R_2)$ we obtain for the renormalized dissipation to second order (with $\ln v \to \ln v/\omega_c$ implied below)

$$\frac{1}{\eta^R_2} = \frac{1}{\eta} - \frac{\ln v}{\eta^2} + \frac{\ln^2 v - \ln v}{\eta^3} + \delta\left(\frac{1}{\eta^R_2}\right). \tag{55}$$

The contribution of the last term is peculiar and depends on the order in which the limits are taken. We define a nonequilibrium limit where η^R is evaluated for a strictly dc field (i.e., $\omega \to 0$ is taken first) and then a logarithmically divergent $E \neq 0$ term is obtained; namely,

$$\begin{aligned} \delta\left(\frac{1}{\eta^R_2}\right) &= \frac{1}{\eta^2}\lim_{v\to 0}\lim_{\omega\to 0}\frac{1}{i\omega} \times \int_{t1,t2} R_{t_1}B_{t_1}B_{t_2}\sin vt_1 \sin vt_2(1 - e^{i\omega t_1})t_1 \\ &= -\frac{1}{\eta^3}\lim_{v\to 0}\int_{t_1} R_{t_1}B_{t_1}t_1^2 \sin vt_1 \int_{t_2} R_{t_2}B_{t_2}\sin vt_2 \\ &= \lim_{v\to 0}\frac{1}{\eta^3}\int^\infty \sin(vt_1)\int^\infty \sin(vt_2)/t_2^2 \\ &= \lim_{v\to 0}\frac{1}{\eta^3}\frac{1}{v}v\ln v + O(v) = \frac{1}{\eta^3}\ln v. \end{aligned} \tag{56}$$

Considering next the alternative equilibrium order of limits (i.e., first $E \to 0$), we obtain

$$\lim_{\omega\to 0}\lim_{v\to 0}\sin(vt_1)\sin(vt_2) = 0, \tag{57}$$

hence $\delta(1/\eta^R_2) = 0$. The renormalized η to second order is then

$$\frac{1}{\eta^R_2} = \frac{1}{\eta} - \frac{\ln v}{\eta^2} + \frac{\ln^2 v + b_0\ln v}{\eta^3}, \tag{58}$$

where b_0 depends on the order of limits, the nonequilibrium case has $b_0 = 0$, while the equilibrium one has $b_0 = -1$. The latter case is in fact the known equilibrium result.[16] The distinction between the two limits will become more pronounced in the full quantum treatment.

B. Numerical solution of Langevin equation

We solve the nonlinear Langevin equation numerically. The time is discretized to $t = T/N \times (1,2,\ldots,N)$, with T being the total time span of system. The noise term ξ^i_t is generated numerically using a discrete Fourier transform of $\xi^i_\omega = \sqrt{B_\omega T}\mathcal{R}^i$ where $\mathcal{R}^i$ is a unit white Gaussian noise. The correlation function linearity requires introducing a high-frequency cutoff τ_0. We choose the cutoff to be in Lorentzian form $B_\omega = \hbar\eta|\omega|/(1 + \omega^2\tau_0^2)$, in the following section we explain the importance of this choice.

We solve the equation in an iterative procedure. Using the convolution form

$$\theta_t = \int_{t'} R_{t,t'}\left[\xi^x_{t'}\cos\theta_{t'} - \xi^y_{t'}\sin\theta_{t'} - E\right], \tag{59}$$

starting with an arbitrary configuration of $\theta^{(0)}_t$ we calculate the right-hand side (RHS) of (59) to find a new $\theta^{(1)}_t$. We repeat the procedure n times until the expression is saturated when $\theta^{(n)}_t = \theta^{(n+1)}_t$. This procedure is improved if instead of taking the convolution result as the next order θ_t we use some mixing of that result and of the previous θ_t configuration in the form $\theta^{(m)}_t = (1 - \beta)\theta^{(m-1)}_t + \beta \times \text{RHS}$ where β is a mixing parameter. Typically n would be of the order of 10^5 and $\beta = 0.1$.

With this choice the Langevin equation takes the following form:

$$\begin{aligned} m\ddot{\theta}_t &= \xi^x_t\cos\theta_t + \xi^y_t\sin\theta_t + E + \Delta_t, \\ \Delta_t &= \frac{\eta}{\tau_0^2}\int_{-\infty}^t \sin[\theta_t - \theta_{t'}]e^{-(t-t')/\tau_0}dt', \end{aligned} \tag{60}$$

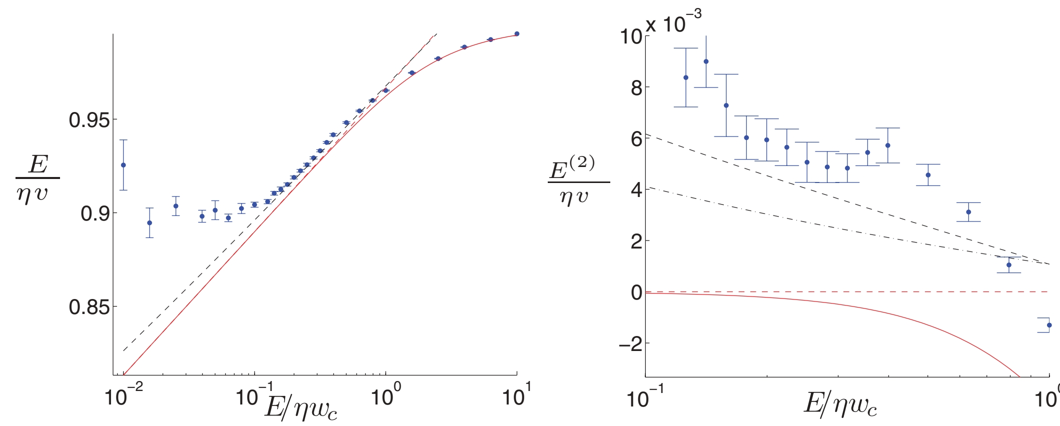

FIG. 2. (Color online) (Left panel) Velocity-field relation for Eq. (60) with $\eta = 30\hbar/\pi$, $\omega_c = 100/\tau_0$, and $\tau_0 = 20\Delta\tau$. Here $N = 2^{15}$, $\Delta\tau = 1/20$. The circles are numerical data, the full red line is a first-order perturbation in $1/\eta$, the dashed lower red line is its logarithmic expansion for large $\ln v/\omega_c$ and the dashed upper (black) line includes the second-order logarithmic term, corresponding to Eq. (58) for $b_0 = 0$. Note that the data is not reliable for $E/(\eta\omega_c) \lesssim 1/(\Delta\tau N\omega_c) \sim 0.06$. (Right panel) The same data and line types after subtracting the first-order terms [i.e., $E^{(2)}/(\eta v) = E/(\eta v) - 1 - \hbar(\ln(v/\omega_c) - 1)/(\pi\eta)$]. An additional dash-dotted line corresponds to $b_0 = -1$, which is a worse fit to the data than $b_0 = 0$ (dashed upper line). Note that the numerical data displays E/v rather than dE/dv, hence Eq. (53) acquires a -1 term.

where Δ_t is a correction term defined by δR_ω^{-1} in the response function (19) because $\int_{t'} \delta R_{t,t'}^{-1}[\xi_{t'}^x \cos\theta_{t'} + \xi_{t'}^y \sin\theta_{t'} + E] = -\int_\omega m\omega^2 \Delta_\omega$.

In the numerical system we have now four time scales, two numerical time scales, i.e. the time segment $\Delta\tau = \bar{T}/N$ and the time span $\bar{T}$, as well as the two physical high-frequency cutoffs, $1/\tau_0$ for the noise and ω_c the mass cutoff. The region of interest corresponds to velocity $v^R = \langle\dot{\theta}_t\rangle$ between the time scales $\Delta\tau \ll \tau_0 < 1/\omega_c \ll 1/v^R \sim 1/v < \bar{T}$. The inequality $\tau_0 < 1/\omega_c$ is useful since we compare the numerical result to an asymptotic result in which ω_c rather than $1/\tau_0$ is the high-frequency cutoff.

With the result for θ_t we can find the renormalized $1/\eta^R = dv^R/dE$ with $v^R = \langle\dot{\theta}_t\rangle$ where the average $\langle\ldots\rangle$ reflects an average on both the time domain $t > 1/\omega_c$ and on numerous realizations of the noise.

In the left panel of Fig. 2 our numerical solution for the Langevin equation is shown, including a fit to the second order with $b_0 = 0$. On the right panel the first order is subtracted with either the nonequilibrium $b_0 = 0$ or the equilibrium $b_0 = -1$. The first is in fact a better fit for the numerical data. When $1/v$ approaches the simulation time span $\bar{T}$ the numerics become unreliable, as the particle cannot complete even one revolution in time $\bar{T}$; a plateau is then observed at low E.

With the numerical results for θ_τ we can also generate the correlation function $\tilde{C}_\tau = \langle[\theta_\tau - \theta_0]^2\rangle$, the first-order perturbation for this correlation function is given in Eq. (49). In Fig. 3 we plot this correlation function as a function of the time separation τ for the same parameters as in Fig. 2, with and without a finite field. The data is fairly close to the first-order result (49) for not too long times; that is, for zero field the correlation has a subdiffusion logarithmic behavior while for finite force the correlation has a diffusion ($\sim\tau$) behavior.

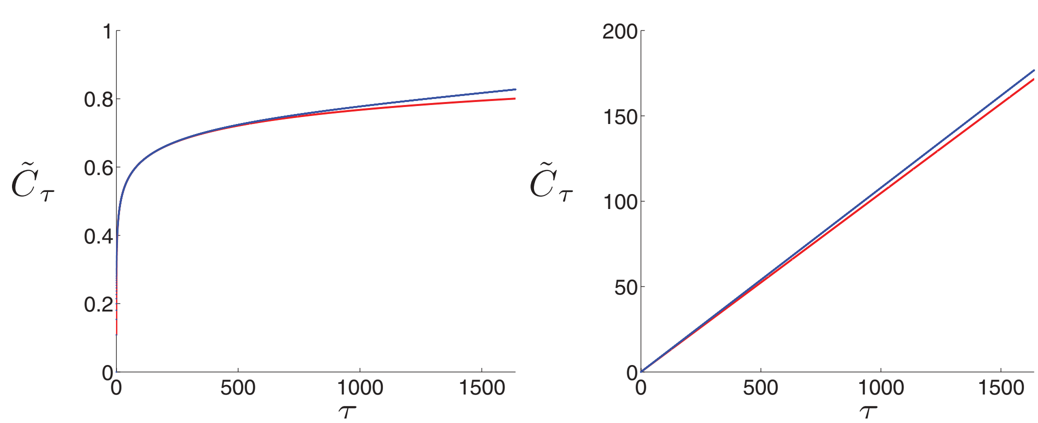

FIG. 3. (Color online) (Left panel) Correlation function $\tilde{C}_\tau$ as a function of time (blue, upper line) and the asymptotic results of Eq. (49) (red, lower line) for $E = 0$. (Right panel) Correlation function as a function of time (blue, upper line) and the asymptotic results of Eq. (49) (red, lower line) for $E/\eta = 1$ and $\tau_0 = 1$.

IV. QUANTUM RENORMALIZATION GROUP

A. Perturbations from S_{int}

Consider now the definition η^R in Eqs. (28) and (29):

$$-i\delta E^{(1)} = \left\langle \frac{\delta S_{\text{int}}}{\delta\hat{\theta}_t}\right\rangle_0 = \frac{2}{\hbar}\int_{t'} B_{t,t'}\left\langle \cos\left(\frac{\hbar}{2}\hat{\theta}_t\right)\sin\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right)\cos(vt - vt' + \delta\theta_t - \delta\theta_{t'})\right\rangle_0$$
$$= \frac{2}{\hbar}\int_{t'} B_{t,t'}\sum_{\sigma,\sigma',\mu=\pm}\frac{\sigma'}{8i}\left\langle \exp\left[\frac{1}{2}i\hbar\sigma\hat{\theta}_t + \frac{1}{2}i\hbar\sigma'\hat{\theta}_{t'} + i\mu(vt - vt' + \delta\theta_t - \delta\theta_{t'})\right]\right\rangle_0$$
$$= \frac{2}{\hbar}\int_{t'} B_{t,t'}\sum_{\sigma,\sigma',\mu=\pm}\frac{\sigma'}{8i}\exp\left[-\frac{1}{2}\mu\hbar(\sigma i R_{t't} - \sigma' i R_{tt'}) + i\mu(vt - vt')\right]. \tag{61}$$

For $t < t'$ the term $\sigma' R_{tt'} = 0$ and then $\sum \sigma' = 0$. The result is then finite only for $t > t'$; defining $\mu' = \mu\sigma'$,

$$= \frac{2}{\hbar}\int_{t'} B_{t,t'}\sum_{\sigma',\mu'=\pm}\frac{\sigma'}{4i}\exp\left[i\sigma'\mu'(vt - vt') + \frac{1}{2}i\hbar\mu' R_{tt'}\right] = i\frac{2}{\hbar}\int_{t'} B_{t,t'}\sin v(t-t')\sin\left(\frac{1}{2}\hbar R_{tt'}\right). \tag{62}$$

Hence the force correction is

$$\delta E^{(1)} = -\frac{2}{\hbar}\int_{\tau} B_\tau \sin\left(\frac{1}{2}\hbar R_\tau\right)\sin(v\tau), \tag{63}$$

so that, by using Eq. (29) and performing the calculation of the integrals with arbitrary cutoffs τ_0 and $\omega_c^{-1} = m/\eta$, one obtains

$$\frac{1}{\eta^R} = \frac{1}{\eta} - \frac{2}{\pi\eta}\left[\sin\left(\frac{\hbar}{2\eta}\right)\ln(v/\omega_c) + C + \mathcal{O}(v)\right], \tag{64}$$

where the constant C depends on τ_0 and ω_c. Although we will not need it below, its detailed form is given in Appendix C in the limit $\tau_0 = 0$.

Consider next second order in S_{int}:

$$i\delta E^{(2)} = \frac{1}{2}\left\langle\frac{\delta}{\delta\hat{\theta}_{t_1}}S_{\text{int}}^2\right\rangle = \frac{1}{2}4\left(\frac{2}{\hbar^2}\right)^2\frac{\hbar i}{2\times 2^6}\sum_{\epsilon_i\sigma,\sigma'=\pm}\epsilon_2\epsilon_3\epsilon_4\int_{t_2,t_3,t_4} B_{t_1,t_2}B_{t_3,t_4}e^{i\sigma v(t_1-t_2)+i\sigma' v(t_3-t_4)}$$
$$\times\left\langle \exp\left[\frac{1}{2}i\left(\epsilon_1\hat{\theta}_{t_1} + \epsilon_2\hat{\theta}_{t_2} + \epsilon_3\hat{\theta}_{t_3} + \epsilon_4\hat{\theta}_{t_4}\right) + i\sigma\left(\theta_{t_1} - \theta_{t_2}\right) + i\sigma'\left(\theta_{t_3} - \theta_{t_4}\right)\right]\right\rangle_0. \tag{65}$$

Note that $\delta/\delta\hat{\theta}_{t_1}$ can be applied also at either t_2,t_3,t_4 and all these terms are identical since $\sin(\frac{1}{2}\hbar\hat{\theta}_{t_i})$ appears in the same form for all t_i, hence a factor of four. Now change all $\epsilon_i, \sigma, \sigma' \to -(\epsilon_i, \sigma, \sigma')$ and define $\sigma' = \sigma\mu$ to obtain

$$i\delta E^{(2)} = \frac{i^2}{16\hbar^3}\sum_{\epsilon_i\sigma,\mu=\pm}\epsilon_2\epsilon_3\epsilon_4\sigma\int_{t_2,t_3,t_4} B_{t_1,t_2}B_{t_3,t_4}\sin[v(t_1-t_2) + \mu v(t_3-t_4)]$$
$$\times\exp\left\{-\frac{1}{2}\hbar\left\langle\sigma\left(\epsilon_1\hat{\theta}_{t_1} + \epsilon_2\hat{\theta}_{t_2} + \epsilon_3\hat{\theta}_{t_3} + \epsilon_4\hat{\theta}_{t_4}\right)\left[\theta_{t_1} - \theta_{t_2} + i\mu\left(\theta_{t_3} - \theta_{t_4}\right)\right]\right\rangle_0\right\}$$
$$= \frac{-1}{8\hbar^3}\sum_{\epsilon_i,\mu=\pm}\epsilon_2\epsilon_3\epsilon_4\int_{t_2,t_3,t_4} B_{t_1,t_2}B_{t_3,t_4}A_2\sin[v(t_1-t_2) + \mu v(t_3-t_4)], \tag{66}$$

where

$$A_2 = \exp\left\{\tfrac{1}{2}i\hbar\epsilon_1\left(-R_{t_2,t_1} + \mu R_{t_3,t_1} - \mu R_{t_4,t_1}\right)\right\}\exp\left\{\tfrac{1}{2}i\hbar\epsilon_2\left(R_{t_1,t_2} + \mu R_{t_3,t_2} - \mu R_{t_4,t_2}\right)\right\}$$
$$\times\exp\left\{\tfrac{1}{2}i\hbar\epsilon_3\left(R_{t_1,t_3} - R_{t_2,t_3} - \mu R_{t_4,t_3}\right)\right\}\exp\left\{\tfrac{1}{2}i\hbar\epsilon_4\left(R_{t_1,t_4} - R_{t_2,t_4} + \mu R_{t_3,t_4}\right)\right\}. \tag{67}$$

Note that in A_2 if t_2 is the maximal time then its second factor equals 1 and $\sum_{\epsilon_2}\epsilon_2 = 0$. Similarly, if t_3 (or t_4) is the maximal time, the third (or fourth) factor equals 1 and $\sum_{\epsilon_3}\epsilon_2 = 0$ (or $\sum_{\epsilon_4}\epsilon_2 = 0$). Therefore, t_1 must be the maximal time and the first factor equals 1. The result is symmetric in $t_3 \leftrightarrow t_4$, so choose $t_3 > t_4$, with factor two. Hence three time orderings, denoted by A,

B, C, $\delta E^{(2)} = \delta E_A + \delta E_B + \delta E_C$,

$$\begin{aligned}
\delta E_A &= \frac{4}{\hbar^3}\sum_\mu \int_{t_1>t_2>t_3>t_4} \sin\left(\frac{1}{2}\hbar R_{t_1,t_2}\right)\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_3}-R_{t_2,t_3}\right)\right]\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_4}-R_{t_2,t_4}+\mu R_{t_3,t_4}\right)\right]\\
&\quad\times B_{t_1,t_2}B_{t_3,t_4}\sin[v(t_1-t_2)+\mu v(t_3-t_4)],\\
\delta E_B &= \frac{4}{\hbar^3}\sum_\mu \int_{t_1>t_3>t_2>t_4} \sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_2}+\mu R_{t_3,t_2}\right)\right]\sin\left(\frac{1}{2}\hbar R_{t_1,t_3}\right)\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_4}-R_{t_2,t_4}+\mu R_{t_3,t_4}\right)\right]\\
&\quad\times B_{t_1,t_2}B_{t_3,t_4}\sin[v(t_1-t_2)+\mu v(t_3-t_4)],\\
\delta E_C &= \frac{4}{\hbar^3}\sum_\mu \int_{t_1>t_3>t_4>t_2} \sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_2}+\mu R_{t_3,t_2}-\mu R_{t_4,t_2}\right)\right]\sin\left(\frac{1}{2}\hbar R_{t_1,t_3}\right)\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_4}+\mu R_{t_3,t_4}\right)\right]\\
&\quad\times B_{t_1,t_2}B_{t_3,t_4}\sin[v(t_1-t_2)+\mu v(t_3-t_4)].
\end{aligned} \tag{68}$$

The B and C terms can be time ordered as A by $t_2 \leftrightarrow t_3$ in B and $t_2 \to t_4$, $t_4 \to t_3$, $t_3 \leftrightarrow t_2$ in C. In terms of the $\mu = \pm$ components,

$$\begin{aligned}
\delta E_A^+ + \delta E_C^- &= \frac{4}{\hbar^3}\int_A \sin\left(\frac{1}{2}\hbar R_{t_1,t_2}\right)\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_3}-R_{t_2,t_3}\right)\right]\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_4}-R_{t_2,t_4}+R_{t_3,t_4}\right)\right]\\
&\quad\times\left[B_{t_1,t_2}B_{t_3,t_4}+B_{t_1,t_4}B_{t_2,t_3}\right]\sin[v(t_1-t_2+t_3-t_4)],\\
\delta E_A^- + \delta E_B^- &= \frac{4}{\hbar^3}\int_A \sin\left(\frac{1}{2}\hbar R_{t_1,t_2}\right)\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_3}-R_{t_2,t_3}\right)\right]\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_4}-R_{t_2,t_4}-R_{t_3,t_4}\right)\right]\\
&\quad\times\left[B_{t_1,t_2}B_{t_3,t_4}+B_{t_1,t_3}B_{t_2,t_4}\right]\sin[v(t_1-t_2+t_4-t_3)],\\
\delta E_B^+ + \delta E_C^+ &= \frac{4}{\hbar^3}\int_A \sin\left(\frac{1}{2}\hbar R_{t_1,t_2}\right)\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_3}+R_{t_2,t_3}\right)\right]\sin\left[\frac{1}{2}\hbar\left(R_{t_1,t_4}-R_{t_3,t_4}+R_{t_2,t_4}\right)\right]\\
&\quad\times\left[B_{t_1,t_3}B_{t_2,t_4}+B_{t_1,t_4}B_{t_2,t_3}\right]\sin[v(t_1-t_3+t_2-t_4)].
\end{aligned} \tag{69}$$

In Appendix E we derive the $\ln^2 v$ coefficient directly for the single-cutoff case where $\tau_0 = 0$. Here we proceed with a shorter indirect method. In general we have two cutoffs m/η, τ_0 in Eq. (7) and we define $\tau_1(m/\eta,\tau_0)$ as the cutoff time for the response R_t [Eq. (7)]. For the purpose of identifying the leading $\ln^2 v$ term we take a formal limit such that this cutoff time is $\tau_1 \to 0$. We will eventually restore physical cutoffs corresponding to m/η, τ_0 in R_t. The only cutoff for now is τ_0 in B_ω [Eq. (6)]. In this limit $R_t \to \frac{1}{\eta}\Theta(t)e^{-\delta t}$ where $\delta \to +0$ to ensure the retarded nature [poles of $1/(\omega + i\delta)$]. The significant virtue of this limit is that the first two equations of (69) vanish since $R_{t_1,t_3} - R_{t_2,t_3} \to 0$, leaving just the last form. The evaluation of $\delta E^{(2)}$ in this limit is straightforward (Appendix D), leading to

$$\delta E^{(2)} = \frac{4\eta^2}{\pi^2\hbar}\sin^2\left(\frac{\hbar}{2\eta}\right)\sin\left(\frac{\hbar}{\eta}\right)v\ln(v\tau_0)[\ln(v\tau_0)+1]. \tag{70}$$

Hence, from Eq. (29),

$$\frac{1}{\eta^{R(2)}} = \frac{4}{\pi^2\hbar}\sin^2\left(\frac{\hbar}{2\eta}\right)\sin\left(\frac{\hbar}{\eta}\right)[\ln^2(v\tau_0)+3\ln(v\tau_0)+1]. \tag{71}$$

So far $\delta E^{(2)}$ is calculated in a formal limit $\tau_1 \to 0$. We proceed by asserting that for any τ_0,τ_1 the leading singularity as $v \to 0$ is a $\ln^2 v$ term, as expected for a two-loop calculation. This term must involve an η-dependent function $f_\eta(\tau_0,\tau_1)$ that has dimensions of time. Fixing the coefficient of $\ln^2[v f_\eta(\tau_0,\tau_1)]$ as in Eq. (71), we have $f_\eta(\tau_0,0) = \tau_0$ while for $\tau_0 \to 0$, when $\tau_1 \to m/\eta = 1/\omega_c$ we must have the form $f_\eta(0,\tau_1) = b(\eta)\tau_1 = b(\eta)/\omega_c$. The two-loop correction (71) becomes, at $\tau_0 = 0$,

$$\frac{1}{\eta^{R(2)}} = \frac{4}{\pi^2\hbar}\sin^2\left(\frac{\hbar}{2\eta}\right)\sin\left(\frac{\hbar}{\eta}\right)\ln^2\left[\frac{v}{\omega_c}b(\eta)\right] + O(\ln v). \tag{72}$$

The renormalized friction therefore has the form

$$\frac{1}{\eta^R} = \frac{1}{\eta} - \frac{2}{\pi\eta}\sin\left(\frac{\hbar}{2\eta}\right)\ln\left[\frac{v}{\omega_c}\right] + \frac{4}{\pi^2\hbar}\sin^2\left(\frac{\hbar}{2\eta}\right)\sin\left(\frac{\hbar}{\eta}\right)\times\left\{\ln^2\left[\frac{v}{\omega_c}\right] + b_0(\eta)\ln\left[\frac{v}{\omega_c}\right]\right\}. \tag{73}$$

We have thus identified the coefficient of the $\ln^2(v)$ term; this coefficient is also identified by the more lengthy calculation of the $\tau_0 = 0$ case in Appendix E. In Appendix E we furthermore show that the coefficient of the $\ln v$ term [i.e., $\sin^2(\hbar/(2\eta))\sin(\hbar/\eta)b_0(\eta)$], has at least one factor of $\sin(\hbar/(2\eta))$. Hence the perturbation expansion as well as the following RG analysis are justified near the zeros of $\sin(\hbar/(2\eta))$.

We note that in the semiclassical limit the perturbation expansion is in $R^{2n-1}B^n/\eta^2 \sim 1/\eta^{n+1}$ for large η; in the quantum case the R^{2n-1} factors become periodic functions. The main conclusion is that there is a new small parameter in the perturbation series, $\sin(\hbar/(2\eta))$.

B. Perturbations from S_c

Here we consider the S_c interaction in Eq. (21). The S_c terms are

$$\langle \hat{\theta}_{t'} \theta_t S_c \rangle = \langle \hat{\theta}_{t'} \theta_t S_c^2 \rangle = 0. \tag{74}$$

However, the mixed term and the corresponding correction to $1/\eta$ are

$$\begin{aligned} \delta R^m_{t,t'} &= i \langle \hat{\theta}_{t'} \theta_t S_c S_{\rm int} \rangle, \\ \Rightarrow \frac{1}{\eta^m} &= \frac{2}{\pi \hbar}\left[\sin\frac{\hbar}{2\eta}\left(\sin\frac{\hbar}{\eta} - \frac{\hbar}{\eta}\right)\right. \\ &\quad \left. + \frac{\hbar}{2\eta}\cos\frac{\hbar}{2\eta}\left(\sin\frac{\hbar}{\eta} - \frac{\hbar}{\eta}\right)\right] \ln(\upsilon\tau_1), \end{aligned} \tag{75}$$

which does not vanish at $\sin(\hbar/(2\eta)) = 0$. Note, however that this term is $\sim\hbar^3$ (i.e., a three-loop term). Furthermore, other response functions do show such zeros. For example, for the $\bar{R}_{t,t'}$ correlation [Eq. (77) below] we have $\langle \theta_t \sin\frac{\hbar}{2}\hat{\theta}_{t'} S_c\rangle = 0$ to first order, while in second order,

$$\begin{aligned} \delta \bar{R}^m_{t,t'} &= \frac{2i}{\hbar}\left\langle \theta_t \sin\frac{\hbar}{2}\hat{\theta}_{t'} S_c S_{\rm int}\right\rangle, \\ \Rightarrow \frac{1}{\bar{\eta}^m} &= \frac{2}{\pi\hbar}\sin\frac{\hbar}{\eta}\left(\sin\frac{\hbar}{\eta} - \frac{\hbar}{\eta}\right)\ln(\upsilon\tau_1). \end{aligned} \tag{76}$$

We note that there are many other operators that have vanishing perturbations at $\sin(\hbar/(2\eta)) = 0$ to second order in $S_{\rm int}$, S_c; for example, the dissipation term in Eq. (9) $\langle \theta_t \sin(\hbar\hat{\theta}_{t'})\rangle$, or the response to an ac field with frequency υ $\langle \theta_t \cos\delta\theta_{t'} \sin\frac{\hbar}{2}\hat{\theta}_{t'}\rangle$.

C. Renormalization-group analysis

We note that in Eq. (73) $g = (2/\pi)\sin(\hbar/(2\eta))$ acts as an unexpected small parameter for the expansion, since all divergences vanish when $g = 0$. It raises the interesting possibility that $g = 0$ be viewed as a RG fixed point. For that we need to find a renormalized coupling which obeys multiplicative RG, the simplest choice being $g_R = (2/\pi)\sin(\hbar/[2\eta^R(E)])$. The question is then whether the β function $\beta = -E\partial_E g_R$ can be written only in terms of g_R. Although the nonperiodic $1/\eta$ factor in Eq. (73) appears at first to be problematic, we propose that resummation from higher loops, which allows for higher-order terms $O(1/\eta^4)$ changes the one-loop term in Eq. (73) by $\hbar/(2\eta) \to \sin(\hbar/(2\eta))$.

To further motivate this proposal we consider the response

$$\bar{R}_{t,t'} = i\frac{2}{\hbar}\left\langle \theta_t \sin\left(\frac{\hbar}{2}\hat{\theta}_{t'}\right)\right\rangle. \tag{77}$$

Physically, $\exp(\pm i\hbar\hat{\theta}_{t'}/2)$ corresponds to an electric field pulse $\delta E(t) = \pm(\hbar/2)\delta(t-t')$ or equivalently a rapid change of flux by $\pm 1/2$, therefore $\bar{R}_{t,t'}$ corresponds to the difference in response to these two flux pulses. Defining the dissipation parameter $\bar{\eta}^R$ for $\bar{R}_{t,t'}$ as in Eq. (25) we obtain that the one-loop term is fully periodic with

$$\frac{\hbar}{2\bar{\eta}^R} = \frac{\hbar}{2\eta} - \frac{2}{\pi}\sin^2\left(\frac{\hbar}{2\eta}\right)\ln[\tau_1\upsilon], \tag{78}$$

hence $\hbar/(2\eta) \to \sin(\hbar/(2\eta))$ in Eq. (73).

We propose then that a RG consistent theory corresponds to

$$\begin{aligned} \frac{\hbar}{2\eta^R} &= \frac{\hbar}{2\eta} - \frac{2}{\pi}\sin^2\left(\frac{\hbar}{2\eta}\right)\ln[\tau_1\upsilon] + \frac{4}{\pi^2}\sin^3\left(\frac{\hbar}{2\eta}\right) \\ &\quad \times \cos\left(\frac{\hbar}{2\eta}\right)\{\ln^2[\tau_1\upsilon] + b_0(\eta)\ln[\tau_1\upsilon]\}. \end{aligned} \tag{79}$$

Taking a sine of both sides it yields to order g^3, with $b_0 = b_0(g=0)$,

$$g_R = g \mp g^2\ln(\upsilon/\omega_c) + g^3[\ln^2(\upsilon/\omega_c) + b_0\ln(\upsilon/\omega_c)], \tag{80}$$

where $\pm$ refers to $g = 0$ with $\cos(\hbar/(2\eta)) = \pm 1$, leading to

$$\beta(g_R) = \frac{dg^R}{-d\ln\upsilon} = \pm g_R^2 - b_0 g_R^3 + O(g_R^4). \tag{81}$$

This RG equation is satisfied for both $\pm$ fixed points as seen by substituting Eq. (80). We propose then that $g^R = 0$ are exact zeros of the perturbation expansion and the additional requirement of a RG structure leads to the result (80).

Equation (80) yields fixed points at $\hbar/(2\eta_n) = n\pi$ with $n = 1,2,3,\ldots$ that are attractive at $\eta > \eta_n$ and repulsive at $\eta < \eta_n$ (i.e., the flow of $\eta \neq \eta_n$ is always to smaller η). At these fixed points a Gaussian evaluation yields the correlation $\langle\cos\theta_t\cos\theta_0\rangle \sim t^{-2n}$. We recall now a theorem for the lattice model[31] where the equilibrium action with mass-related cutoff is replaced by an action on a lattice resulting in an XY model with long-range interactions. The theorem states[31] that $\langle\cos\theta_t\cos\theta_0\rangle \sim 1/t^2$; this result was also derived[9] to first order in η. The range $\eta > \eta_1$ has a RG flow to η_1 and is therefore consistent with the theorem. The hypothesis of Gaussian fixed points corresponding to $n \geqslant 2$ is inconsistent with the theorem; that is, $\langle\cos\theta_t\cos\theta_0\rangle$ becomes a relevant operator at the $n \leqslant 2$ points rendering them unstable. Note that in the SEB problem $\cos\theta_t$ corresponds to a lead-dot voltage and its correlations determine the SET conductance,[11,13,21] while in the ring problem it corresponds to fluctuations in the circular asymmetry.

For $\eta < \eta_1$ the system could have non-Gaussian fixed points or a line of fixed points as hinted by the small η perturbation.[9] The equilibrium $K_1(\phi_x)$ was evaluated for small η and for $T \to 0$ has the form[22] $K_1(\phi_x) \sim \delta(\phi_x - 1/2)/T$ (i.e., the dissipation is concentrated at the single point $\phi_x = 1/2$). This implies from Eq. (39) that $\eta^R \sim T$ and therefore vanishes at temperature $T = 0$. It is not clear, however, that $\eta = 0$ is a fixed point in the RG sense and if so what is its range of attraction. An $\eta = 0$ fixed point would imply the implausible result that the ring conductance diverges for small but finite η. We therefore expect that $\eta_1 \equiv \eta^R$ is the single fixed point in this system, as illustrated in Fig. 4.

V. DISCUSSION

The special value $\eta^R = \hbar/(2\pi)$ has a topological interpretation as a Thouless charge pump,[26] as shown in the introduction.

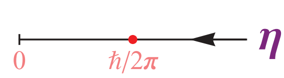

FIG. 4. (Color online) RG flow of η.

Hence a slow change in ϕ_x by one unit results in transporting a unit charge once around the ring if $\eta^R = \hbar/(2\pi)$. Such quantization has been shown for cases where the spectrum has a gap,[26] although quantized charge transport was shown also in cases without a gap;[32,33] in our case the gap vanishes[15] at flux $\phi_x = 1/2$. Vanishing of this gap is essential in solving for the dissipation problem in the ring via Landau-Zener transitions, as studied in related models.[34] We note that the quantized η^R also results from arguing that there should be a unique frequency $\omega_E = (2\pi/\hbar)E = v$ as $E \to 0$ [see discussion below Eq. (38)], as suggested by linear response.

We conclude from Eq. (45) that for $\eta > \eta_1 \equiv \eta^R$ the SEB satisfies the quantization (see definitions in Sec. II E)

$$\int_0^1 \frac{C_0^2(N_0)}{C_g^2} R_q(N_0) dN_0 = \frac{h}{e^2}. \tag{82}$$

In particular, when $\eta/\hbar \gtrsim 1$ we have[6–9] from the known $M^*/m \sim e^{\pi\eta/\hbar}$ and from Eq. (6) $C_0/C_g = 1 + O(e^{-\pi\eta/\hbar})$. We expect R_q to be independent of N_0 at large η, hence

$$R_q = \frac{h}{e^2}[1 + O(e^{-\pi\eta/\hbar})], \tag{83}$$

similar to the $N_c = 1$ case.[3]

The conductance of the ring can be defined by the voltage around the ring $2\pi E/e$ and the current $e\langle\dot\theta\rangle/(2\pi)$, hence we predict that the conductance for $\eta > \eta^R$ is

$$G_{\text{ring}} = \frac{e^2}{4\pi^2\eta^R} = \frac{e^2}{h}. \tag{84}$$

While this well-known quantum conductance seems natural, we emphasize that it is due to the inherent nonequilibrium nature of the driving force and the specific limiting procedure of taking a dc limit before the linear response limit [Eq. (39)].

Finally, we consider the conditions for our proposed box experiment. The Coulomb box (i.e., a metallic quantum dot) should be connected to the electrode with $N_c \gg 1$ degenerate channels; in fact N_c can be fairly small and yet reproduce the $N_c \to \infty$ case, except at exponentially small temperatures.[35] By analogy with $E = \hbar\dot\phi_x$ in the ring, we propose measuring the response to a gate voltage that is linear in time $N_0 = Et$. This leads to a dc current into the Coulomb box whose dissipation is the average in Eq. (45). The field E should be sufficiently small so that g_R is sufficiently near the fixed point. For an initial $g \approx 1$ integration of $\partial g_R/\partial \ln E = g_R^2$ yields $g_R = 1/\ln(\hbar\omega_c/E) \ll g$. For example, for $g_R \lesssim 0.1$ and a typical $\hbar\omega_c \approx 1$ meV one needs $E/\hbar \lesssim 10^8$ Hz. $E/\hbar$ has frequency units, corresponding to 10^8 electrons/s flowing into the box.

While it may be possible to measure dissipation directly (e.g., via heating), we propose measuring instead the charge fluctuations (noise) $S_Q(\omega) = e^2\langle\hat N_t \hat N_{t'}\rangle_\omega$. The latter should be measured at frequency, temperature, and level spacings Δ such that $\Delta < \omega$, $T \ll 10^8$ Hz, to yield the response to the force E. FDT relates the (symmetrized) noise and the retarded response $K(\omega)$ [Eq. (41)] via $S_Q(\omega) = \hbar \coth[\hbar\omega/(2T)]\text{Im}K(\omega)$. From Eq. (45) we have (at $T = 0$) that the gate voltage averaged noise $\bar S_Q(\omega)$ satisfies $\bar S_Q(\omega)(\frac{2E_c}{e\hbar})^2/\omega = \hbar/\eta^R$. In particular, as the fixed point is approached we predict $\bar S_Q(\omega)(\frac{2E_c}{e\hbar})^2/\omega = 2\pi$.

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APPENDIX A: MAPPING COULOMB BOX AND RING

The AES mapping has been extensively used, yet we find it useful to reproduce it since the relation between correlation functions has received less attention.

The Coulomb-box action corresponding to the Hamiltonian (40) is

$$\begin{aligned} -i\hbar S &= \int_t \left\{ \sum_\alpha d^\dagger_{\alpha,i}(i\hbar\partial_t - \epsilon_\alpha)d_{\alpha,i} - E_c(\hat N - N_0)^2 \right\} - i\hbar S_{\text{lead}} - i\hbar S_{\text{tunn}}, \\ -i\hbar S_{\text{lead}} &= \int_t \sum_k a^\dagger_{k,i}(i\hbar\partial_t - \epsilon_k)a_{k,i}, \quad -i\hbar S_{\text{tunn}} = \int_t \sum_{k,\alpha} t_{k,\alpha,i} a^\dagger_{k,i} d_{\alpha,i} + \text{H.c.}, \end{aligned} \tag{A1}$$

with the partition $Z = e^{-S}$. Adding a variable $\dot\theta_t$ to the path integral yields

$$\begin{aligned} -i\hbar S &= \int_t \left\{ E_c\left[\hat N - N_0 - \frac{\hbar}{2E_c}\dot\theta_t\right]^2 + \sum_\alpha d^\dagger_{\alpha,i}(i\hbar\partial_t - \epsilon_\alpha)d_{\alpha,i} - E_c(\hat N - N_0)^2 \right\} - i\hbar S_{\text{lead}} - i\hbar S_{\text{tunn}} \\ &= \int_t \left\{ \sum_\alpha d^\dagger_{\alpha,i}(i\hbar\partial_t - \epsilon_\alpha - \hbar\dot\theta_t)d_{\alpha,i} + \frac{1}{4E_c}[\hbar\dot\theta_t + 2E_c N_0]^2 \right\} - i\hbar S_{\text{lead}} - i\hbar S_{\text{tunn}}. \end{aligned} \tag{A2}$$

Now define $d_\alpha = e^{-i\theta_t}\tilde{d}_\alpha$:

$$-i\hbar S = \int_t \left\{ \sum_\alpha \tilde{d}^\dagger_{\alpha,i}(i\hbar\partial_t - \epsilon_\alpha)\tilde{d}_{\alpha,i} + \frac{\hbar^2}{4E_c}\dot{\theta}_t^2 + \dot{\theta}_t N_0 + \sum_{k,\alpha,i}[t_{k,\alpha,i}a^\dagger_{k,i}\tilde{d}_{\alpha,i}e^{i\theta_t} + \text{H.c.}] \right\} - i\hbar S_{\text{lead}}. \tag{A3}$$

The ring action in terms of θ_t is derived by integrating out the fermions $\tilde{d}_\alpha$ and a_k. Define time-ordered Greens' functions on the dot $G_{0\alpha,i}(\omega) = (\omega - \epsilon_{\alpha,i} + i\,\text{sgn}\omega 0^+)^{-1}$ and on the lead $G_{0k,i}(\omega) = (\omega - \epsilon_{k,i} + i\,\text{sgn}\omega 0^+)^{-1}$. In matrix notation,

$$\hat{G}_i^{-1}(t,t') = \begin{pmatrix} G^{-1}_{0\alpha,i}(t,t') & 0 \\ 0 & G^{-1}_{0k,i}(t,t') \end{pmatrix} + \begin{pmatrix} 0 & t_{k,\alpha,i}e^{i\theta_t} \\ t^*_{k,\alpha,i}e^{-i\theta_t} & 0 \end{pmatrix}\delta(t-t') \equiv \hat{G}_{0i}^{-1} + \hat{T}_i. \tag{A4}$$

The trace over fermions, using $\det(iG) = \exp(\text{Tr}\ln iG)$, yields

$$S_{\text{eff}} = -\sum_i \text{Tr}\ln i\hat{G}_i^{-1}(t,t) = -\sum_i \text{Tr}\ln\left\{i\hat{G}_{0i}^{-1}(t,t')[\delta(t-t') + \hat{G}_{0i}(t',t)\hat{T}_i(t)]\right\}. \tag{A5}$$

Expanding in $\hat{T}$, the zeroth order is θ_t independent, the first order vanishes, hence to second order,

$$S_{\text{eff}} = -\frac{1}{2}\sum_i \text{Tr}\{\hat{G}_0\hat{T}\hat{G}_0\hat{T}\} = -\frac{1}{2}\sum_i \int_{t,t'} G_{0\alpha,i}(t,t')G_{0k,i}(t',t)|t_{k,\alpha,i}|^2 e^{i\theta_t - i\theta_{t'}} + \text{H.c.} \tag{A6}$$

For completeness we derive the Matsubara effective action using $\sum_\alpha G_{\alpha,i}(\tau) = T\sum_n G(\omega_n)e^{i\omega_n\tau}$ with fermionic $\omega_n = \pi T(2n+1)$:

$$\begin{aligned} G(\omega_n) &= \int_\epsilon \frac{\rho_{\text{dot}}(\epsilon)}{i\omega_n - \epsilon} = \int_0^\infty \rho_{\text{dot}}(\epsilon)\left[\frac{1}{i\omega_n - \epsilon} + \frac{1}{i\omega_n + \epsilon}\right] = \int_0^\infty \rho_{\text{dot}}(\epsilon)\frac{-2i\omega_n}{\omega_n^2 + \epsilon^2} = -i\pi\rho_{\text{dot}}(0)\,\text{sgn}(\omega_n), \\ \sum_\alpha G_{0\alpha,i}(\tau) &= 2\pi\rho_{\text{dot}}(0)\sum_{n>0}\sin(\omega_n\tau) = \rho_{\text{dot}}(0)\frac{\pi T}{\sin(\pi T\tau)}, \end{aligned} \tag{A7}$$

where $\rho_{\text{dot}}(\epsilon)$ is the dot density of states, assumed symmetric, and eventually constant. With the lead density of states $\rho_{\text{lead}}(\epsilon)$, and assuming a constant $t_{k,\alpha,i}$,

$$S_{\text{eff}} = -\frac{1}{2}|t|^2 N_c\rho_{\text{dot}}(0)\rho_{\text{lead}}(0)\iint \frac{\pi^2 T^2}{\sin^2[\pi T(\tau - \tau')]}\cos[\theta(\tau) - \theta(\tau')], \tag{A8}$$

where $N_c = \sum_i$ is the number of channels. This is the well-known equilibrium ring system with a bosonic CL environment,[6–9] where $\eta = \frac{1}{2}\pi|t|^2 N_c\rho_{\text{dot}}(0)\rho_{\text{lead}}(0)$ and $m = 1/(2E_c)$. The expansion in $\hat{T}$ is justified for $|t|^2 \to 0$; however, with $N_c \to \infty$ any value of η can be generated. In fact N_c can be fairly small and yet reproduce the $N_c \to \infty$ case, except at exponentially small temperatures.[35] A similar derivation holds for the Keldysh action leading to the form (21).

We proceed now to map observables of the Coulomb box to those of the ring problem. Since the action (A3) has a term $+\dot{\theta}N_0$ we identify $N_0 = -\phi_x$ where ϕ_x is the flux through the ring (in units of the quantum flux). Hence,

$$\begin{aligned} \hbar\langle\dot{\theta}\rangle &= \int_\theta \hbar\dot{\theta}\exp\left[-\frac{i}{\hbar}\int E_c\left(\hat{N} - N_0 - \frac{\hbar}{2E_c}\dot{\theta}\right)^2 + \text{fermion terms}\right] \\ &= \int_\theta (\hbar\dot{\theta} + 2E_c\hat{N} - 2E_cN_0)\exp\left[-\frac{i}{\hbar}\int \frac{\hbar^2}{4E_c}\dot{\theta}^2 + \text{fermion terms}\right] = 2E_c[\langle\hat{N}\rangle - N_0]. \end{aligned} \tag{A9}$$

In particular, without interaction, $t_{k\alpha} = 0$, the charge has no fluctuations $\langle\hat{N}\rangle = 0$ (for $|N_0| < \frac{1}{2}$) so that $\hbar\langle\dot{\theta}\rangle = -2E_cN_0 = 2E_c\phi_x$.

Consider next the time-ordered $\mathcal{T}$ correlations (the following is the same for $\langle\dot{\theta}_t^+\dot{\theta}_{t'}^+\rangle$, $\langle\dot{\theta}_t^+\dot{\theta}_{t'}^-\rangle$ with $\pm$ Keldysh contours),

$$\begin{aligned} \hbar^2\mathcal{T}\langle\dot{\theta}_t\dot{\theta}_{t'}\rangle &= \int_{\tilde{\theta}} \hbar^2\dot{\theta}_t\dot{\theta}_{t'}\exp\left[-\frac{i}{\hbar}\int E_c\left(\hat{N} - N_0 - \frac{\hbar^2}{2E_c}\dot{\theta}\right)^2 + \text{fermion terms}\right] \\ &= \int_\theta (\hbar\dot{\theta}_t + 2E_c\hat{N}_t - 2E_cN_0)(\hbar\dot{\theta}_{t'} + 2E_c\hat{N}_{t'} - 2E_cN_0)\exp\left[-\frac{i}{\hbar}\int \frac{\hbar^2}{4E_c}\dot{\tilde{\theta}}^2 + \text{fermion terms}\right] \\ &= \hbar^2\mathcal{T}\langle\dot{\theta}_t\dot{\theta}_{t'}\rangle_0 + 4E_c^2\mathcal{T}\langle(\hat{N}_t - N_0)(\hat{N}_{t'} - N_0)\rangle. \end{aligned} \tag{A10}$$

To obtain the retarded response,

$$-i\mathcal{D}^R_{t,t'} = \theta(t-t')\langle[A_t,B_{t'}]\rangle = \theta(t-t')\langle A_tB_{t'} - B_{t'}A_t\rangle = \mathrm{T}\langle A^+_tB^+_{t'}\rangle - \langle B^-_{t'}A^+_t\rangle, \tag{A11}$$

where $\pm$ are Keldysh contour indices, so that A^+ is earlier than B^-.

Define the response $K_{t,t'}$ of the Coulomb box, as well as the response of ring problem $\tilde{K}_{t,t'}$ in the form (displayed here with operators whose $\langle A_t\rangle = 0$ to allow relation with time ordering)

$$\tilde{K}_{t,t'} = +i\theta(t-t')\langle[(\dot{\theta}_t - \langle\dot{\theta}\rangle),(\dot{\theta}_{t'} - \langle\dot{\theta}\rangle)]\rangle, \quad K_{t,t'} = +i\theta(t-t')\langle[(\hat{N}_t - \langle\hat{N}\rangle),(\hat{N}_{t'} - \langle\hat{N}\rangle)]\rangle. \tag{A12}$$

From Eq. (A10) we have

$$\hbar^2\mathcal{T}\langle(\dot{\theta}_t - \langle\dot{\theta}\rangle)(\dot{\theta}_{t'} - \langle\dot{\theta}\rangle)\rangle + \hbar^2\langle\dot{\theta}\rangle^2 = \hbar^2\mathcal{T}\langle\dot{\theta}_t\dot{\theta}_{t'}\rangle_0 + 4E_c^2\mathcal{T}\langle(\hat{N}_t - \langle\hat{N}\rangle)(\hat{N}_{t'} - \langle\hat{N}\rangle)\rangle + 4E_c^2\big(\langle\hat{N}\rangle^2 - 2N_0\langle\hat{N}\rangle + N_0^2\big). \tag{A13}$$

Now using Eq. (A9) and that the relation (A10) holds for both terms in Eq. (A11), a relation between these response functions is obtained

$$\hbar^2\tilde{K}_{t,t'} = -2E_c\hbar\delta(t-t') + 4E_c^2K_{t,t'}, \tag{A14}$$

which is reproduced as Eq. (40). This relation is consistent with results in Ref. 22.

APPENDIX B: SEMICLASSICAL CASE: FIRST AND SECOND ORDER

1. First-order term

First-order perturbation of the Green's function

$$\begin{aligned} R^{(1)}_{t,t'} &= -i\frac{1}{2}\int_{t_1,t_2} B_{t_1,t_2}\big\langle\hat{\theta}_{t_1}\hat{\theta}_{t_2}\cos\big(\theta_{t_1}-\theta_{t_2}\big)\hat{\theta}_{t'}\theta_t\big\rangle_{S_0} \\ &= \frac{-i}{4}\int_{t_1,t_2} B_{t_1,t_2}\sum_{\sigma=\pm}\partial_{\alpha_{i=1,2,3,4}}\exp\big(i\alpha_1\hat{\theta}_{t_1} + i\alpha_2\hat{\theta}_{t_2} + i\sigma\theta_{t_1} - i\sigma\theta_{t_2} + i\alpha_3\hat{\theta}_{t'} + i\alpha_4\theta_t\big)\Big|_{\alpha_i=0}. \end{aligned} \tag{B1}$$

An averaging with Gaussian weight

$$\langle e^{i\theta_{t_1}+i\theta_{t_2}+\cdots+i\hat{\theta}_{t_1}+i\hat{\theta}_{t_2}+\cdots}\rangle = e^{i\langle\theta_{t_1}+\theta_{t_2}+\cdots\rangle}e^{-\langle(\theta_{t_1}+\theta_{t_2}+\cdots)(\hat{\theta}_{t_1}+\hat{\theta}_{t_2}+\cdots)\rangle} = e^{ivt_1+ivt_2+\cdots}e^{iR_{t_1,t_2}+iR_{t_2,t_1}+\cdots}. \tag{B2}$$

The retarded function

$$\begin{aligned} R^{(1)}_{t,t'} &= \frac{1}{4i}\int_{t_1,t_2}\sum_{\sigma=\pm}\partial_{\alpha_i}B_{t_1,t_2}\exp\big[i\alpha_1\big(-\sigma R_{t_2,t_1} + \alpha_4R_{t,t_1}\big) + i\alpha_2\big(\sigma R_{t_1,t_2} - \alpha_4R_{t,t_1}\big) + i\alpha_3\big(\sigma R_{t_1,t'} - \sigma R_{t_2,t'} + \alpha_4R_{t,t_1}\big)\big]e^{i\sigma v(t_1-t_2)} \\ &= \frac{1}{4}\int_{t_1,t_2}\sum_{\sigma=\pm}\partial_{\alpha_4}B_{t_1,t_2}\big(\sigma R_{t_2,t_1} - \alpha_4R_{t,t_1}\big)\big(\sigma R_{t_1,t_2} + \alpha_4R_{t,t_1}\big)\big(\sigma R_{t_1,t'} - \sigma R_{t_2,t'} + \alpha_4R_{t,t_1}\big)e^{i\sigma v(t_1-t_2)} \\ &= -\int_{t_1,t_2}B_{t_1,t_2}\cos v(t_1-t_2)R_{t,t_1}R_{t_1,t_2}\big(R_{t_1,t'} - R_{t_2,t'}\big). \end{aligned} \tag{B3}$$

In the last expression we use $R_tR_{-t} = 0$.

2. Second-order term

Using the same procedure for the second order:

$$\begin{aligned} R^{(2)}_{t,t'} &= \frac{i}{2}\langle\hat{\theta}_{t'}\theta_t(S_{\text{int}})^2\rangle = -\frac{i}{8}\int_{t_1,t_2,t_3,t_4}B_{t_1,t_2}B_{t_3,t_4}\big\langle\hat{\theta}_{t_1}\hat{\theta}_{t_2}\cos\big(\theta_{t_1}-\theta_{t_2}\big)\hat{\theta}_{t_3}\hat{\theta}_{t_4}\cos\big(\theta_{t_3}-\theta_{t_4}\big)\hat{\theta}_{t'}\theta_t\big\rangle \\ &= \frac{1}{2^5i}\int_{t_1,\ldots,4}B_{t_1,t_2}B_{t_3,t_4}\sum_{\sigma_1,\sigma_2=\pm}\partial_{\alpha_{i=1,\ldots,6}}\big\langle\exp\big(i\alpha_1\hat{\theta}_{t_1} + i\alpha_2\hat{\theta}_{t_2} + i\alpha_3\hat{\theta}_{t_3} + i\alpha_4\hat{\theta}_{t_4} + i\sigma_1\theta_{t_1} - i\sigma_1\theta_{t_2} + i\sigma_2\theta_{t_3} \\ &\quad - i\sigma_2\theta_{t_4} + i\alpha_3\hat{\theta}_{t'} + i\alpha_4\theta_t\big)\big\rangle\big|_{\alpha_i=0}, \end{aligned} \tag{B4}$$

using the symmetry between $\sigma_1 \leftrightarrow -\sigma_1$ and $t_1 \leftrightarrow t_2$ and similarly for t_3,t_4:

$$\begin{aligned} R^{(2)}_{t,t'} &= \frac{1}{8}\int_{t_1,t_2,t_3,t_4}B_{t_1,t_2}B_{t_3,t_4}e^{iv(t_1-t_2)-iv(t_3-t_4)}\partial_{\alpha_6}\big[-R_{t_2,t_1} + R_{t_3,t_1} - R_{t_4,t_1} + \alpha_6R_{t,t_1}\big]\big[R_{t_1,t_2} + R_{t_3,t_2} - R_{t_4,t_2} + \alpha_6R_{t,t_2}\big] \\ &\quad\times\big[R_{t_1,t_3} - R_{t_2,t_3} - R_{t_4,t_3} + \alpha_6R_{t,t_3}\big]\big[R_{t_1,t_4} - R_{t_2,t_4} + R_{t_3,t_4} + \alpha_6R_{t,t_4}\big]\big[R_{t_1,t'} - R_{t_2,t'} + R_{t_3,t'} - R_{t_4,t'} + \alpha_6R_{t,t'}\big], \end{aligned} \tag{B5}$$

the choice $t_1 > t_2, t_3, t_4$, only R_{t,t_1} remains. R_τ is real, we separate the exponent into two sine and two cosine terms as follows:

$$R^{(2)}_{t,t'} = \frac{1}{8}\int_{t_1,t_2,t_3,t_4} B_{t_1,t_2}B_{t_3,t_4}[\cos v(t_1-t_2)\cos v(t_3-t_4) - \sin v(t_1-t_2)\sin v(t_3-t_4)]R_{t,t_1}\big[R_{t_1,t_2}+R_{t_3,t_2}-R_{t_4,t_2}\big]$$
$$\times\big[R_{t_1,t_3}-R_{t_2,t_3}-R_{t_4,t_3}\big]\big[R_{t_1,t_4}-R_{t_2,t_4}+R_{t_3,t_4}\big]\big[R_{t_1,t'}-R_{t_2,t'}+R_{t_3,t'}-R_{t_4,t'}\big]. \tag{B6}$$

This long multiplicity of R_t terms is now separated into eight different terms. For the terms with the cosine we calculate explicitly three terms, which we label by a to c. Term "a" is

$$R^a_{t,t'} = \frac{1}{2}\int_{t_1,t_2,t_3,t_4} B_{t_1,t_2}\cos v(t_1-t_2)R_{t,t_1}R_{t_1,t_2}\big(R_{t_1,t'}-R_{t_2,t'}\big)B_{t_3,t_4}\cos v(t_3-t_4)\big(R_{t_1,t_3}-R_{t_2,t_3}\big)\big(R_{t_1,t_4}-R_{t_2,t_4}\big)$$
$$= \frac{1}{2}\int_{t_1,t_2} B_{t_1,t_2}\cos v(t_1-t_2)R_{t,t_1}R_{t_1,t_2}\big(R_{t_1,t'}-R_{t_2,t'}\big)\tilde{C}_{t_1,t_2}. \tag{B7}$$

This term in ω space

$$R^a_\omega = -\frac{1}{2}R^2_\omega\int_t R_t B_t \cos vt\ (e^{i\omega t}-1)\tilde{C}_t,$$

with $\tilde{C}_t = 2(C^{(1)}_{t=0} - C^{(1)}_t)$. Similarly we choose two different terms "b" and "c" and write them directly in ω space:

$$R^b_\omega = R^2_\omega\int_t R^{(1)}_t B_t\cos vt(e^{i\omega t}-1), \tag{B8}$$

$$R^c_\omega = R^3_\omega\left[\int_t R_t B_t\cos vt(e^{i\omega t}-1)\right]^2 = R^{-1}_\omega\big(R^{(1)}_\omega\big)^2. \tag{B9}$$

Note the $R^{(1)}_t$ in the expression R^b is the first-order result of the retarded Green function. R^c_ω is the reducible term containing multiplication of $R^{(1)}_\omega$. Renormalized η for small v is

$$\frac{1}{\eta^a_2} = \frac{1}{2}\frac{1}{\eta^2}\int_t R_t B_t\tilde{C}(t)t = \frac{\hbar}{\pi\eta^3}\int_t R_t B_t t[\ln t+\gamma+O(v)+O(1/t)] = -\frac{\hbar^2}{2\pi^2\eta^3}\ln^2 v + O(v),$$
$$\frac{1}{\eta^b_2} = -\frac{\hbar}{\pi\eta^2}\int_t R^{(1)}_t B_t t = -\frac{\hbar}{\pi\eta^3}\int_t R_t B_t t[\ln t+\gamma+1+O(v)+O(1/t)] = \frac{\hbar^2}{2\pi^2\eta^3}\ln^2 v - \frac{\hbar^2}{2\pi^2\eta^3}\ln v + O(v), \tag{B10}$$
$$\frac{1}{\eta^c_2} = \frac{1}{\eta^3}\left[\int_t R_t B_t t\right]^2 = \frac{\hbar^2}{2\pi^2\eta^3}[\ln v + O(v)]^2 = \frac{\hbar^2}{2\pi^2\eta^3}\ln^2 v + O(v).$$

The terms containing the sine in Eq. (B6) are, in general, of order $O(v)$; however, we have identified the following term which, depending on the order of limits, may contribute a term logarithmic in v for small v:

$$R^d_\omega = -R^2_\omega\int_{t_1,t_2} R_{t_1}R_{t_2}B_{t_1}B_{t_2}\sin vt_1\sin vt_2(1-e^{i\omega t_1})\int_{t_3}\big(R_{t_1+t_3}-R_{t_3}\big). \tag{B11}$$

We label the dissipation parameter form this term by $\delta(1/\eta^R_2) = \lim_{\omega\to 0}(-i\omega)R^d_\omega$ and find the logarithmic prefactor in Eq. (56), where we use for $t_1 > 0$

$$\int_{t_3}\big(R_{t_1+t_3}-R_{t_3}\big) = \frac{1}{\eta}\int_{-t_1}^{0}\big(1-e^{-(t_1+t_3)\frac{\eta}{m}}\big) + \frac{1}{\eta}\int_0^\infty\big(e^{-t_3\frac{\eta}{m}} - e^{-(t_1+t_3)\frac{\eta}{m}}\big) = \frac{t_1}{\eta}. \tag{B12}$$

APPENDIX C: QUANTUM CASE: FIRST ORDER, MORE DETAILS

Let us give the detailed calculation of the first-order correction in the case of a mass-only cutoff (i.e., $\tau_0 = 0$). Taking the derivative of Eq. (63) in the text we have

$$\partial_v\delta E^{(1)} = -\frac{2}{\hbar}\int_{\tau>0}\tau B(\tau)\sin\left(\frac{\hbar}{2}R(\tau)\right)\cos(v\tau) = \frac{2\eta}{\pi}\int_{\tau>0}\frac{d\tau}{\tau}\sin\left[\frac{\hbar}{2\eta}\big(1-e^{-\frac{\eta}{m}\tau}\big)\right]\cos(v\tau)$$
$$= \frac{2\eta}{\pi}\left[\sin\left(\frac{\hbar}{2\eta}\right)\int_{\tau>0}\frac{d\tau}{\tau}\big(1-e^{-\frac{\eta}{m}\tau}\big)\cos(v\tau) - \int_{\tau>0}\frac{d\tau}{\tau}\left\{\sin\left[\frac{\hbar}{2\eta}\big(1-e^{-\frac{\eta}{m}\tau}\big)\right] - \sin\left(\frac{\hbar}{2\eta}\right)\big(1-e^{-\frac{\eta}{m}\tau}\big)\right\}\cos(v\tau)\right]$$
$$= \frac{2\eta}{\pi}\left[\sin\left(\frac{\hbar}{2\eta}\right)\ln\left(\frac{\eta}{mv}\right) + f\left(\frac{\hbar}{2\eta}\right) + \mathcal{O}(v)\right], \tag{C1}$$

since the first integral can be computed exactly and in the second one we can set $v = 0$ to get the constant piece. This determines the constant $C = f(\hbar/(2\eta))$ given in the text in Eq. (64), where the function $f(x)$ is defined as

$$f(x) = \int_0^{+\infty} \frac{dt}{t}\{\sin[x(1-e^{-t})] - \sin(x)(1-e^{-t})\} = -\int_0^1 \frac{dz}{(1-z)\ln(1-z)}[\sin(xz) - z\sin x] = \frac{1}{6}x^3 \ln\left(\frac{8}{3}\right) + O(x^5) \tag{C2}$$

and is a nicely convergent integral, where one can rescale t freely. Although it is not periodic in x, upon plotting it one notes that it seems to become almost periodic at large x.

APPENDIX D: QUANTUM CASE: SECOND ORDER FOR $\tau_1 \to 0$

Since $\sin(\frac{1}{2}\hbar R_{t_1,t_2})$ is a retarded function, we use for $R_t = \Theta(t)e^{-\delta t}$

$$\sin\left(\frac{1}{2}\hbar R_{t_1,t_2}\right) \to \sin\left(\frac{\hbar}{2\eta}\right)e^{-\delta(t_1-t_2)}, \quad \sin\left[\frac{1}{2}\hbar\big(R_{t_1,t_3} + R_{t_2,t_3}\big)\right] \to \sin\left(\frac{\hbar}{\eta}\right)e^{-\delta(t_1-t_3)-\delta(t_2-t_3)},$$
$$\sin\left[\frac{1}{2}\hbar\big(R_{t_1,t_4} - R_{t_3,t_4} + R_{t_2,t_4}\big)\right] \to \sin\left(\frac{\hbar}{2\eta}\right)e^{-\delta(t_1-t_4)-\delta(t_3-t_4)-\delta(t_2-t_4)}. \tag{D1}$$

For example, the Fourier transform of $t_1 - t_3$ and $t_2 - t_3$ should have $1/[(\omega_1 + i\delta)(\omega_2 + i\delta)]$. Define the variables

$$t_2' = t_2 - t_1, \quad t_3' = t_3 - t_2, \quad t_4' = t_4 - t_3, \quad \Rightarrow t_2 = t_2' + t_1, \quad t_3 = t_3' + t_2' + t_1, \quad t_4 = t_4' + t_3' + t_2' + t_1. \tag{D2}$$

These variables are more convenient since their range is independent $-\infty < t_2', t_3', t_4' < 0$. The product of all convergence factors is then $e^{\delta(3t_2'+4t_3'+3t_4')}$, with the factors 3, 4, and 3 unimportant since $\delta \to 0$. Hence

$$\begin{aligned}
\delta E^{(2)} &= \frac{4}{\hbar^3}\sin^2\left(\frac{\hbar}{2\eta}\right)\sin\left(\frac{\hbar}{\eta}\right)\int_{\omega_1,\omega_2} B_{\omega_1}B_{\omega_2}\sum_{\sigma=\pm}\frac{\sigma}{2i}\int_A e^{i\sigma v(-2t_3'-t_4'-t_2')}[e^{i\omega_1(t_3'+t_2')+i\omega_2(t_4'+t_3')} + e^{i\omega_1(t_4'+t_3'+t_2')+i\omega_2 t_3'}]e^{\delta(t_2'+t_3'+t_4')}\\
&= \frac{4}{\hbar^3}\sin^2\left(\frac{\hbar}{2\eta}\right)\sin\left(\frac{\hbar}{\eta}\right)\int_{\omega_1,\omega_2} B_{\omega_1}B_{\omega_2}\sum_{\sigma=\pm}\frac{\sigma}{2i}\left[\frac{1}{-i\sigma v + i\omega_2 + \delta} + \frac{1}{-i\sigma v + i\omega_1 + \delta}\right]\\
&\quad\times\frac{1}{(-2i\sigma v + i\omega_1 + i\omega_2 + \delta)(-i\sigma v + i\omega_1 + \delta)}\\
&= \frac{4}{\hbar^3}\sin^2\left(\frac{\hbar}{2\eta}\right)\sin\left(\frac{\hbar}{\eta}\right)\sum_\sigma \frac{\sigma}{2}(\hbar\eta)^2\int\frac{d\omega_1}{2\pi}\frac{1}{(\omega_1 - \sigma v - i\delta)^2}\frac{|\omega_1|}{1+\omega_1^2\tau_0^2}\int\frac{d\omega_2}{2\pi}\frac{1}{\omega_2 - \sigma v - i\delta}\frac{|\omega_2|}{1+\omega_2^2\tau_0^2},
\end{aligned} \tag{D3}$$

where the integral over ω_2 gives

$$\begin{aligned}
\int_0^\infty d\omega_2\left[\frac{1}{\omega_2 - \sigma v - i\delta} - \frac{1}{-\omega_2 - \sigma v - i\delta}\right]\frac{\omega_2}{1+\omega_2^2\tau_0^2} &= 2\sigma v\int_0^\infty d\omega_2 \frac{\omega_2}{(\omega_2^2 - v^2)(1+\omega_2^2\tau_0^2)}\\
&= -\sigma v\ln(v\tau_0) + O\big(v^3\tau_0^2\ln(v\tau_0)\big),
\end{aligned} \tag{D4}$$

and over ω_1 gives

$$\begin{aligned}
&\int_0^\infty d\omega_1\left[\frac{1}{(\omega_1 - \sigma v - i\delta)^2} + \frac{1}{(-\omega_1 - \sigma v - i\delta)^2}\right]\frac{\omega_1}{1+\omega_1^2\tau_0^2}\\
&\quad= 2\int_0^\infty d\omega_1\left[\frac{\omega_1}{\omega_1^2 - v^2} + \frac{2v^2\omega_1}{(\omega_1 - \sigma v - i\delta)^2(\omega_1 + \sigma v + i\delta)^2}\right] = -2\ln(v\tau_0) - 2,
\end{aligned} \tag{D5}$$

where in the last integral $\tau_0 \to 0$ can be taken. Substituting (D4) and (D5) into (D3) leads to the result (70).

APPENDIX E: QUANTUM CASE: SECOND ORDER WITH MASS CUTOFF

In this Appendix we rederive the second-order quantum case using directly a mass cutoff. In particular, we identify the coefficient of the $\ln^2 v$ term, confirming that coefficient in Eq. (70), and derive some properties of the second-order $\ln v$ term.

We express Eq. (66) as

$$\delta E^{(2)} = \frac{i}{4\hbar^3}\sum_{\epsilon_i}\epsilon_2\epsilon_3\epsilon_4\int_{t_2,t_3,t_4} B_{t_1,t_2}B_{t_3,t_4}A_2\sin[v(t_1 - t_2 + v(t_3 - t_4)], \tag{E1}$$

where the symmetry between t_3 and t_4 is used to sum over $\mu = \pm$. Defining $F_{t_i,t_j} = \exp(i\epsilon\hbar R_{t_i,t_j}/2) - 1$ Eq. (67) can be expressed as

$$\begin{aligned} A_2 &= \left(F_{t_2,t_1}^{-\epsilon_1}+1\right)\left(F_{t_3,t_1}^{\epsilon_1}+1\right)\left(F_{t_4,t_1}^{-\epsilon_1}+1\right)\left(F_{t_1,t_2}^{\epsilon_2}+1\right)\left(F_{t_3,t_2}^{\epsilon_2}+1\right)\left(F_{t_4,t_2}^{-\epsilon_2}+1\right)\\ &\quad\times\left(F_{t_1,t_3}^{\epsilon_3}+1\right)\left(F_{t_2,t_3}^{-\epsilon_3}+1\right)\left(F_{t_4,t_3}^{-\epsilon_3}+1\right)\left(F_{t_1,t_4}^{\epsilon_4}+1\right)\left(F_{t_2,t_4}^{-\epsilon_4}+1\right)\left(F_{t_3,t_4}^{\epsilon_4}+1\right)\\ &= \left(F_{t_1,t_2}^{\epsilon_2}+F_{t_2,t_1}^{-\epsilon_1}+1\right)\left(F_{t_1,t_3}^{\epsilon_3}+F_{t_3,t_1}^{\epsilon_1}+1\right)\left(F_{t_1,t_4}^{\epsilon_4}+F_{t_4,t_1}^{-\epsilon_1}+1\right)\left(F_{t_3,t_2}^{\epsilon_2}+F_{t_2,t_3}^{-\epsilon_3}+1\right)\left(F_{t_4,t_2}^{-\epsilon_2}+F_{t_2,t_4}^{-\epsilon_4}+1\right)\left(F_{t_4,t_3}^{-\epsilon_3}+F_{t_3,t_4}^{\epsilon_4}+1\right). \end{aligned} \tag{E2}$$

In the last expression we used the retarded property of F_{t_i,t_j} so that $F_{t_i,t_j}F_{t_j,t_i} = 0$. When transforming all functions to their frequency domain

$$\delta E^{(2)} = \frac{1}{16\hbar^3}\int_{\omega_a,\omega_b}\left(\left[B_{\omega_a+v}+B_{\omega_a-v}\right]\left[B_{\omega_b+v}-B_{\omega_b-v}\right]+\left[B_{\omega_a+v}-B_{\omega_a-v}\right]\left[B_{\omega_b+v}+B_{\omega_b-v}\right]\right)K(\omega_a,\omega_b), \tag{E3}$$

$$\begin{aligned} K(\omega_a,\omega_b) &= \sum_{\epsilon_i}\epsilon_2\epsilon_3\epsilon_4\int_{\omega_1,..\omega_6}\left[F_{\omega_1}^{\epsilon_2}+F_{-\omega_1}^{-\epsilon_1}+2\pi\delta(\omega_1)\right]\left[F_{\omega_2}^{\epsilon_3}+F_{-\omega_2}^{\epsilon_1}+2\pi\delta(\omega_2)\right]\left[F_{\omega_3}^{\epsilon_4}+F_{-\omega_3}^{-\epsilon_1}+2\pi\delta(\omega_3)\right]\\ &\quad\times\left[F_{\omega_4}^{\epsilon_2}+F_{-\omega_4}^{-\epsilon_3}+2\pi\delta(\omega_4)\right]\left[F_{\omega_5}^{-\epsilon_2}+F_{-\omega_5}^{-\epsilon_4}+2\pi\delta(\omega_5)\right]\left[F_{\omega_6}^{-\epsilon_3}+F_{-\omega_6}^{\epsilon_4}+2\pi\delta(\omega_6)\right]\\ &\quad\times(2\pi)^3\delta(\omega_a+\omega_1+\omega_4+\omega_5)\delta(-\omega_b+\omega_2-\omega_4+\omega_6)\delta(\omega_b+\omega_3-\omega_5-\omega_6). \end{aligned} \tag{E4}$$

We notice that the function $K(\omega_a,\omega_b)$ can have poles at $\omega_a,\omega_b = i\delta$ leading to a logarithmic divergence term for either a $O(\omega^{-1})$ term with the antisymmetric expression

$$\int_\omega (B_{\omega+v}-B_{\omega-v})\frac{1}{\omega-i\delta} = -2\int_0^\infty B_\tau\sin(v\tau)d\tau = \frac{2\hbar\eta}{\pi}v\ln(v)+O(v), \tag{E5}$$

or for $O(\omega^{-2})$ terms with the symmetric expression

$$\int_\omega (B_{\omega+v}+B_{\omega-v})\frac{1}{(\omega-i\delta)^2} = -2\int_0^\infty \tau B_\tau\cos(v\tau)d\tau = \frac{2\hbar\eta}{\pi}\ln(v)+O(v), \tag{E6}$$

where $\delta = +0$. Note that the Fourier transform of $1/(\omega-i\delta)$ is $e^{-\delta\tau}\Theta(\tau)$ while that of $1/(\omega-i\delta)^2$ is $e^{-\delta\tau}\tau\Theta(\tau)$. We keep here only the long-time divergence, controlled by $\ln v$. Keeping also short-time divergences would eventually replace $\ln v \to \ln(v/\omega_c)$ with $\omega_c = \eta/m$. Equations (E5) and (E6) show that $\ln^2(v)$ terms arises from either a $1/(\omega_a\omega_b^2)$ or $1/(\omega_a^2\omega_b)$ terms in $K(\omega_a,\omega_b)$.

We use the retarded property of $F_\tau = F_\tau\Theta(\tau)$ and expand the function in powers of $\hbar/\eta$

$$F_\omega^\epsilon = e^{i\epsilon\hbar/(2\eta)}\sum_{n=0}^\infty\frac{1}{n!}\left(-\frac{i\hbar\epsilon}{2\eta}\right)^n\frac{i}{\omega+in\omega_c+i\delta}-\frac{i}{\omega+i\delta}. \tag{E7}$$

Each of the six factors takes the form

$$F_\omega^{\epsilon_i}+F_{-\omega}^{\epsilon_j}+2\pi\delta(\omega) = \sum_{n=0}^\infty\frac{1}{n!}\left(-\frac{i\hbar}{2\eta}\right)^n\left\{\frac{i\epsilon_i^n e^{i\epsilon_i\hbar/(2\eta)}}{\omega+in\omega_c+i\delta}+\frac{i\epsilon_j^n e^{i\epsilon_j\hbar/(2\eta)}}{-\omega+in\omega_c+i\delta}\right\}, \tag{E8}$$

where the delta function cancels with the last terms of the F_ω. We note that $\ln v$ terms arise from terms with at least one vanishing n_j, leading to a pole. For that particular n_j the pole has a coefficient $\exp[i\epsilon_i\hbar/(2\eta)] - \exp[i\epsilon_j\hbar/(2\eta)]$ that vanishes when $\hbar/(2\eta) = \pi\times$ integer. Hence all terms of $\delta E^{(2)}$ have at least one periodic factor of $\sin[\hbar/(2\eta)]$.

The triple-frequency integral Eq. (E4) with the substitution (E8) has 24 terms all with three poles in either ω_a or ω_b. Solving the triple integral and the ϵ_j summations we find

$$\begin{aligned} K(\omega_a,\omega_b) &= \sum_{n_1,\ldots,n_6\geqslant 0}\frac{1}{n_1!n_2!n_3!n_4!n_5!n_6!}\left(-\frac{i\hbar}{2\eta}\right)^{n_1+n_2+n_3+n_4+n_5+n_6}\frac{2}{\omega_c^3}\left\{\frac{[(-1)^{n_2}-(-1)^{n_4}][(-1)^{n_1}-e^{-i\hbar/\eta}]}{(n_2+n_3+n_4+n_5+\delta)(n_1+n_2+n_3+\delta+i\omega_a/\omega_c)}\right.\\ &\quad\times\left(\frac{(-1)^{n_3}-(-1)^{n_5+n_6}e^{i\hbar/\eta}}{n_3+n_5+n_6+\delta-i\omega_b/\omega_c}+\frac{(-1)^{n_5}-(-1)^{n_3+n_6}e^{-i\hbar/\eta}}{n_3+n_5+n_6+\delta i\omega_b/\omega_c}\right)+\frac{(-1)^{n_2}-e^{-i\hbar/\eta}}{(n_1+n_2+n_3+\delta+i\omega_a/\omega_c)}\\ &\quad\times\left(\frac{[(-1)^{n_3}-(-1)^{n_5+n_6}e^{i\hbar/\eta}][(-1)^{n_1}-(-1)^{n_4}]}{(n_3+n_5+n_6+\delta-i\omega_b/\omega_c)[n_1+n_3+n_4+n_6+\delta+i(\omega_a-\omega_b)/\omega_c]}\right.\\ &\quad\left.+\frac{[(-1)^{n_3+n_6}e^{i\hbar/\eta}-(-1)^{n_5}][(-1)^{n_1+n_4}e^{i\hbar/\eta}-e^{-i\hbar/\eta}]}{(n_3+n_5+n_6+\delta+i\omega_b/\omega_c)[n_1+n_3+n_4+n_6+\delta+i(\omega_a+\omega_b)/\omega_c]}\right)\\ &\quad+\frac{(-1)^{n_2}-e^{-i\hbar/\eta}}{(n_1+n_2+n_3+\delta+i\omega_a/\omega_c)(n_1+n_4+n_5+\delta+i\omega_a/\omega_c)}\\ &\quad\left.\times\left(\frac{[(-1)^{n_3}-(-1)^{n_6}][(-1)^{n_1+n_5}e^{i\hbar/\eta}-(-1)^{n_4}]}{[n_1+n_3+n_4+n_6+\delta+i(\omega_a-\omega_b)/\omega_c]}+\frac{[(-1)^{n_1+n_4}e^{i\hbar/\eta}-(-1)^{n_5}][(-1)^{n_3+n_6}e^{i\hbar/\eta}-e^{-i\hbar/\eta}]}{[n_1+n_3+n_4+n_6+\delta+i(\omega_a+\omega_b)/\omega_c]}\right)\right\}. \end{aligned} \tag{E9}$$

At this stage the $\ln^2 v$ term can be simply identified, since this term needs poles in both ω_a and ω_b. The only such term which has the form $[(\omega_a - i\delta)(\omega_b - i\delta)^2]^{-1}$ is the term where $n_1 = n_2 = \cdots = n_6 = 0$; no other term has a zero-frequency divergence at both ω_a and ω_b. For this term we get

$$K_0(\omega_a,\omega_b) = \frac{16\sin^2[\hbar/(2\eta)]\sin(\hbar/\eta)}{(\omega_a - i\delta\omega_c)^2(\omega_b - i\delta\omega_c)}. \tag{E10}$$

And the full expression from Eq. (E1), using Eqs. (E5) and (E6), is then

$$\delta E^{(2)} = \frac{16}{16\hbar^3}\frac{4\hbar^2\eta^2}{\pi^2} v\ln^2(v)\sin^2\frac{\hbar}{2\eta}\sin\frac{\hbar}{\eta} + O(\ln v) = \frac{4\eta^2}{\pi^2\hbar}\sin^2\frac{\hbar}{2\eta}\sin\frac{\hbar}{\eta} v\ln^2(v) + O(\ln v). \tag{E11}$$

This coefficient of the $v\ln^2(v)$ term agrees with that of Eq. (70).

We consider next some of the terms that contribute to the $\ln v$ coefficient. From Eq. (E5) we know that only terms with a single pole [i.e., either $1/(\omega_a - i\delta)$ or $1/(\omega_b - i\delta)$] contribute. We define an expansion

$$K(\omega_a,\omega_b) = K_0(\omega_a,\omega_b) + \sum_{\bar{n}=1}^{\infty}\left(-\frac{i\hbar}{2\eta}\right)^{\bar{n}}\frac{2}{\omega_c^2}k_{\bar{n}}(\omega_a,\omega_b), \tag{E12}$$

where $\bar{n} = \sum_{j=1}^{6} n_j$. Thus there are six terms for $\bar{n} = 1$, 21 terms for $\bar{n} = 2$, and 56 terms for $\bar{n} = 3$. Due to the ω_a, ω_b symmetry we define

$$\kappa_{\bar{n}}(\omega) = \lim_{\omega_a\to 0}\omega_a k_{\bar{n}}(\omega_a,\omega) + \lim_{\omega_b\to 0}\omega_b k_{\bar{n}}(\omega,\omega_b), \tag{E13}$$

so that one integration gives a $\ln v$ while the other gives its coefficient in the form

$$\delta E^{(2)} = \frac{4\eta^2}{\pi^2\hbar}\sin^2\frac{\hbar}{2\eta}\sin\frac{\hbar}{\eta} v\ln^2(v) + \frac{\eta}{2\omega_c^2\hbar^2\pi}\sum_{\bar{n}=1}^{\infty}\left(-\frac{i\hbar}{2\eta}\right)^{\bar{n}}\int_\omega B_\omega\kappa_{\bar{n}}(\omega)v\ln(v) + O(1). \tag{E14}$$

For the first few terms we find

$$\kappa_1(\omega) = \frac{P_2 + P_2\cos\frac{\hbar}{\eta}}{P_4}\sin^2\frac{\hbar}{2\eta}, \quad \kappa_2(\omega) = \frac{P_4 + P_4\cos\frac{\hbar}{\eta}}{P_6}\sin\frac{\hbar}{2\eta}, \quad \kappa_3(\omega) = \frac{P_8 + P_8\cos\frac{\hbar}{\eta}}{P_{10}}\sin^2\frac{\hbar}{2\eta}, \tag{E15}$$

where P_I is a polynomial of ω/ω_c of degree I. The result is consistent with having at least one factor of $\sin[\hbar/(2\eta)]$, as shown above in general.

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