Shot noise in a Majorana fermion chain

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We calculate the shot noise power in a junction of a network of Majorana bound states (MBSs) with a normal metal. These Majorana bound states are on the borders of alternating ferromagnetic and superconducting regions at the quantum spin Hall insulator edge. We analyze different realizations of MBS networks including a few isolated ones and those in a chain allowing for the limits of weak and strong tunneling. The conductance and the shot noise are considerably stronger than those of a weakly coupled normal-superconducting junction, which is a hallmark of the MBS. We find that the Fano factor is quantized, $F = 2$, when one MBS member of a pair is coupled to the lead; however, if both MBS members are coupled F is noninteger.

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Introduction. Recently discovered topological materials in combination with a superconductor can host Majorana fermions.[1–9] Interest in such quasiparticles is due to the possibility of non-Abelian statistics which they satisfy.[10] The most accessible cases are the Majorana bound states (MBSs) at the one-dimensional edge of a quantum spin Hall insulator.[11,12] Two Majorana fermions can form the usual Dirac fermion. However, to detect a state of a Majorana fermion that has no definite charge requires nonlocal measurements. It has been suggested[13,14] that a tunneling probe can detect the presence of MBSs. While the tunneling probe is a local measurement it can detect interference effects between various MBSs. Recently an array of alternating ferromagnetic (FM) and superconducting (SC) regions at the quantum spin Hall (QSH) edge was considered.[15,16] This array may appear due to local density fluctuations at random FM/SC boundaries that host MBSs. The tunneling characteristics (conductance and tunneling current) were calculated for a network of coupled Majorana fermions. Both isolated[16] and random chain[15] cases were analyzed.

In the present work we show how Majorana fermions can be detected by shot noise measurements. We note that previous shot noise evaluations[8,9] have focused on two leads coupled to two MBSs at distinct locations. The experimental search for a MBS focuses on using a scanning tunneling microscope[17] (STM) which was successful in demonstrating chiral spin states. Experimentally it will be much easier to work with a single STM, rather than with two correlated STMs. In our work we focus on the case of a single lead, derive a general expression for an arbitrary configuration of MBSs, and allow for MBS interactions. With a few examples we show that the shot noise is a sensitive probe of the MBS configuration.

The Hamiltonian. We consider the tunneling between a normal metal lead and a realization of a one-dimensional Majorana chain.[12,15,16,18] The coupled Majorana state network is described by the Hamiltonian

$$H_M = \frac{i}{2}\sum_{i,j} t_{ij}\gamma_i\gamma_j, \qquad (1)$$

where γ_i are operators for the ith Majorana fermion, satisfying $\gamma_i = \gamma_i^\dagger$, $\gamma_i^2 = 1$, and t_{ij} are elements of an antisymmetric matrix $\hat{t}$ for the coupling between the i,j MBSs. Disorder may be introduced as random nearest-neighbor coupling. The γ_i can be written as a Bogoliubov transformation for the quasiparticles in the superconductor composed of creation $\Psi_\sigma^\dagger(x)$ and annihilation $\Psi_\sigma(x)$ operators of electrons at position x and with spin σ. The coefficients $f_{\sigma,i}(x)$ in this transformation are the eigenvalues of the Bogoliubov–de Gennes Hamiltonian with zero energy. The general form is

$$\gamma_i = \Sigma_\sigma \int dx[f_{\sigma,i}(x)\Psi_\sigma(x) + f^*_{\sigma,i}(x)\Psi^\dagger_\sigma(x)]. \qquad (2)$$

The tunneling between the normal metal electrode and the superconductor is given by $\mathcal{H}_T = \sum_{k\sigma}\int dx[t_k(x)c^\dagger_{k\sigma}\Psi_\sigma(x) + \text{H.c.}]$. Below we consider the energy gap Δ of the superconductor as the biggest energy scale in the problem. Then for a small applied voltage $eV < \Delta$ only the zero-energy Majorana operator projection of the total quasiparticle operator in the superconductor is important.[15,16,20] Thus the tunneling Hamiltonian and the current operator in Nambu space become

$$H_T = \frac{1}{2}\sum_i \bar{c}(0)\tau_z\hat{V}_i\gamma_i + \text{H.c.}, \qquad (3)$$

$$I = \frac{ie}{2}\sum_i \bar{c}(0)\hat{V}_i\gamma_i + \text{H.c.} \equiv ieJ, \qquad (4)$$

where $c(0) = (c_\uparrow, c_\downarrow, c^\dagger_\downarrow, -c^\dagger_\uparrow)^T$ corresponds to the normal lead electron operator at $x = 0$. The coupling between the normal lead and Majorana states is given by a vector in Nambu space, $\hat{V}_i = (\bar{f}_{\uparrow,i}, \bar{f}_{\downarrow,i}, \bar{f}^*_{\downarrow,i}, -\bar{f}^*_{\uparrow,i})^T$, where $\bar{f}_{\sigma,i} = \int dx f_{\sigma,i}(x)t_k(x)$ describe the interaction of MBSs with the lead electrons and we assumed a weak momentum dependence of matrix element t_k for tunneling to the lead. The Pauli τ matrices act on the $(c_\sigma; \bar{f}_{\sigma,i})$ and $(c^\dagger_\sigma; \bar{f}^*_{\downarrow,i})$ blocks. The total Hamiltonian in addition to parts (1) and (3) includes the lead contribution, which we take at voltage bias V. We write below the action for the lead in the rotated Keldysh basis:

$$S_{\text{lead}} = \frac{1}{2}\int dt\,\Sigma_k \bar{\hat{c}}_k g_k^{-1}\hat{c}_k, \qquad (5)$$

where g_k^{-1} is the inverse Green's function for electrons in the lead. For further use we introduce the Green's function (GF) of the lead integrated over momentum $\bar{g} = \frac{1}{2\pi}\Sigma_k g_k$, which has the following form:

$$\bar{g} = \begin{pmatrix} \bar{g}^R & \bar{g}^K \\ 0 & \bar{g}^A \end{pmatrix}. \qquad (6)$$

Here all entries are 4×4 diagonal matrices in Nambu space. In the energy representation each of them has the form $\bar{g}^{R,A} = \mp\frac{i}{2}N(0)$ diag(1,1,1,1) and $i\bar{g}^K = N(0)$ diag[tanh $\frac{\omega-eV}{2T}(1,1,0,0) +$ tanh $\frac{\omega+eV}{2T}(0,0,1,1)$], where $N(0)$ is the electron density of states in the normal metal lead. To calculate the noise we need to find the effective action as a function of a quantum source. This quantum source consists of the current operator (J) multiplied by source fields $\hat{\lambda} = \mathrm{diag}(\lambda_1, -\lambda_2)$ on the standard (1,2) time Keldysh contour. We obtain a combined source-tunneling contribution to the action:

$$S_s = -\frac{1}{2}\int dt \sum_i [\bar{c}(0)(\tau_z\sigma_z + \hat{I}\hat{\lambda})\hat{V}_i\gamma_i + \gamma_i\hat{V}_i^{\dagger}(\tau_z\sigma_z - \hat{I}\hat{\lambda})c(0)]. \quad (7)$$

Here the Pauli matrix σ_z relates quantum operators [Majorana $\gamma_i = (\gamma_{i1},\gamma_{i2})$ and lead fermions] on the two time contours (1,2). $\hat{I}$ is the unit matrix. Integrating out the lead electron operators and performing a rotation in the Keldysh space in a similar way as we did to obtain Eqs. (5) and (6), we arrive at a simple representation of S_s. Keeping only the quantum source $\lambda_q = (\lambda_1 - \lambda_2)/2$, we arrive at

$$S_s = -\pi \int dt \sum_{i,j} [\gamma_i^{'}\hat{V}_i^{\dagger}(\tau_z\sigma_x - \hat{I}\hat{I}\lambda_q)\bar{g}(\tau_z\hat{I} + \hat{I}\sigma_x\lambda_q)\hat{V}_j\gamma_j^{'}], \quad (8)$$

where $\gamma_i^{'} = (\gamma_i^{\rm cl},\gamma_i^{q})^T = (\gamma_{1i}+\gamma_{2i},\gamma_{1i}-\gamma_{2i})^T/\sqrt{2}$. We also notice a subtlety. The fermion GFs are written in two representations: one, usually used, has the form of Eq. (6); the other is similar to that for boson fields.[19] Due to the self-conjugacy condition the Keldysh GF for Majorana fermions preserves the form of the boson GF:[19]

$$G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix}. \quad (9)$$

The Majorana fermion action follows from Eq. (1) to zero order in tunneling to the lead,

$$S_M = \frac{1}{2}\int dtdt'[\gamma_i^{'}(t)]^T G_{0ij}^{-1}(tt')\gamma_j^{'}(t'); \quad (10)$$

here the inverse matrix GF is

$$\left[G_0^R\right]^{-1} = \tfrac{1}{2}[i\partial_t - 2i\hat{t}]. \quad (11)$$

The total GF follows immediately if we add the zero-source-part contribution of S_s toG_0^{-1}:

$$G^{-1} = G_0^{-1} - 2\pi\hat{V}^{\dagger}\sigma_x\bar{g}\hat{V}, \quad (12)$$

which can be used to verify the form (9). The effective action with the source term becomes

$$S_{\rm eff} = \frac{1}{2}\int dt\gamma'[G^{-1} + Q(\lambda_q)]\gamma', \quad (13)$$

$$Q(\lambda_q) = 2\pi\hat{V}^{\dagger}[\tau_z(\lambda_q\bar{g} - \sigma_x\bar{g}\sigma_x\lambda_q) + \lambda_q\bar{g}\sigma_x\lambda_q]\hat{V}. \quad (14)$$

The partition function for calculations of transport and noise is a function of the quantum field λ_q: $Z(\lambda_q) = \int D\gamma \exp[iS_{\rm eff}]$ and $\ln Z(\lambda_q) = \mathrm{Tr}\ln[1 + GQ(\lambda_q)]$. Taking the variation of $\ln Z$ as $\lambda_q \to 0$, we get expressions for the average current and the noise power:

$$I = \frac{e}{2\hbar}\frac{\delta \ln Z}{\delta\lambda_q} = \frac{e}{2h}\frac{\delta}{\delta\lambda_q}\int d\omega \mathrm{Tr}[GQ], \qquad S_n(tt') = -\frac{1}{2}\frac{\delta^2}{\delta\lambda_q(t)\delta\lambda_q(t')}\ln Z. \quad (15)$$

Before presenting explicit formulas we discuss the approximations which are used: (a) We consider the tunneling element $\bar{f}_{i\sigma}$ as spin independent. (b) When (12) is inverted we neglect off-diagonal $\bar{f}_{i\sigma}\bar{f}_{j\sigma'}$ terms that involve oscillations which vanish upon averaging over large spacings between MBSs.[16] (c) We take the wideband limit, ignoring the momentum dependence of V_i. Thus we have $\Gamma_{\rm ij} = 4\pi N(0)\bar{f}_i^2\delta_{\rm ij}$. (d) We also consider $\Delta > \Gamma_{\rm ij}$. We proceed now to take explicitly the trace in Keldysh and Nambu spaces, leading to general expressions for the current and the total zero-frequency noise power $S_n = S_1 + S_2$:

$$I = \frac{e}{2h}\int d\omega \mathrm{Tr}(\Gamma \mathrm{Im}G^R)\left[\tanh\frac{\omega_-}{2T} - \tanh\frac{\omega_+}{2T}\right],$$
$$S_1 = \frac{2e^2}{h}\int d\omega \mathrm{Tr}(\Gamma \mathrm{Im}G^R)\left[1 - \frac{1}{2}\left(\tanh^2\frac{\omega_-}{2T} + \tanh^2\frac{\omega_+}{2T}\right)\right],$$
$$S_2 = \frac{e^2}{h}\int d\omega \mathrm{Tr}(\Gamma \mathrm{Re}G^R\Gamma \mathrm{Re}G^R)\left(\tanh\frac{\omega_-}{2T} - \tanh\frac{\omega_+}{2T}\right)^2. \quad (16)$$

Here $\omega_{\pm} = \omega \pm eV$, $\mathrm{Im}G^R = (G^R - G^A)/(2i)$ and $\mathrm{Re}G^R = (G^R + G^A)/2$. The term S_1 mainly contributes to the thermal part of the noise power. In the limit of zero temperature $(T \to 0)$, $S_1 \to 0$. The S_2 term for $V > T \to 0$ defines the shot noise, which is the total noise in this temperature limit:

$$S_{\rm shot} = \frac{8e^2}{h}\int_0^{eV} d\omega \mathrm{Tr}(\Gamma \mathrm{Re}G^R\Gamma \mathrm{Re}G^R). \quad (17)$$

Isolated Majorana states. For a single Majorana state the matrix $\hat{t} = 0$ and $G^R = G_{11}^R = 2/(\omega + 2i\Gamma)$. First let us consider the zero-bias conductance in the limit $T > eV \to 0$. In this case S_n corresponds to thermal noise. Directly by calculating the current (16) or with the help of the fluctuation dissipation theorem $\frac{\partial S_n}{4\partial T} = \sigma$, we obtain the linear conductance

$$\sigma = \frac{2e^2}{h}\left(\frac{\Gamma}{T}\right)^2\int_0^{\infty}\frac{dx}{\left[x^2 + \left(\frac{\Gamma}{T}\right)^2\right]\cosh^2 x}. \quad (18)$$

The nonlinear conductance $(V \neq 0)$ at zero temperature is given as[16]

$$\sigma = \frac{2e^2}{h}\frac{4\Gamma^2}{(eV)^2 + 4\Gamma^2}, \quad (19)$$

while for the shot noise of a single Majorana state we obtain the form

$$S_n = \frac{8e^2\Gamma}{h}\left(\arctan\frac{eV}{2\Gamma} - \frac{2eV\Gamma}{(eV)^2 + 4\Gamma^2}\right). \quad (20)$$

More complicated formulas for S_n follow when two MBSs couple via a tunneling element t and couplings Γ_{11},Γ_{22} to the lead. In comparison, the well-known Andreev contribution to

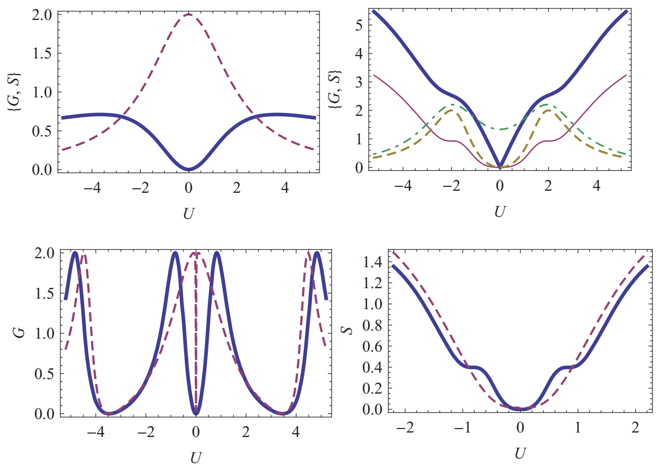

FIG. 1. (Color online) Upper panels: Conductance (dashed or dot-dashed line) and shot noise (full lines). Upper left for a single MBS, upper right for two MBSs with $t = \Gamma$; upper lines (dot-dashed and thick lines) for $\Gamma_{11} = 2\Gamma_{22} = \Gamma$ and lower lines (dashed and thin lines) for $\Gamma_{11} = \Gamma, \Gamma_{22} = 0$. Lower panels four MBSs: Lower left, conductance for weak $t_{12} = 0.1\Gamma$ (dashed line) and strong $t_{12} = \Gamma$ (full line); all $t_{i,i+1}$ are equal and only the first MBS is coupled to the lead. Lower right, the corresponding shot noise. The conductance G is presented in units of e^2/h and the shot noise power S in units of $2\Gamma e^2/h$.

the noise of a normal-superconductor (NS) junction[21] is of order

$$S \sim \frac{4e^3V}{h}\left(\frac{\Gamma}{\Delta}\right)^2. \tag{21}$$

This is much weaker then (20) or the other MBS networks by the factor of $(\Gamma/\Delta)^2 \ll 1$.

Figure 1 presents the conductance and shot noise power as functions of $U = eV/\Gamma$ for different realizations. With two coupled Majoranas the shot noise shows clear steps in its voltage dependence. They corresponds to the peaks in conductance at $eV = \pm 2t$. Therefore a measurement of the noise can detect Majorana fermions. If one of the links is weak, $t_{\rm link} < t$, the noise loses its structure (lower right panel in Fig. 1) and becomes similar to the case of a single Majorana (upper left panel in Fig. 1). In this case the sharp drop of conductance at voltage $eV \to 0$ can be smeared by a small temperature.

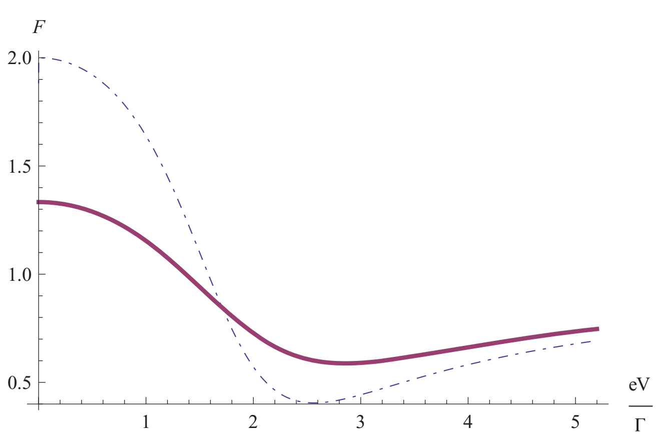

FIG. 2. (Color online) Fano factor F for a double MBS corresponding to the upper right panel of Fig. 1. The dot-dashed curve shows $F(V)$ when only the first MBS is coupled to the lead, while the solid line stands for case when both MBSs couple to the lead: $\Gamma_{11} = \Gamma$ and $\Gamma_{22} = \Gamma/2$.

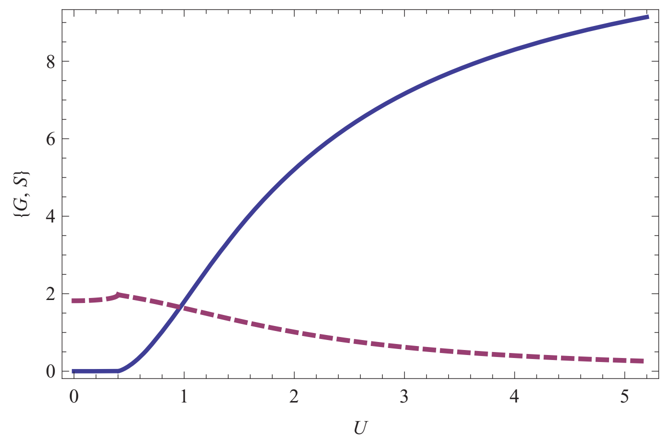

FIG. 3. (Color online) Conductance (dashed curve) and shot noise (solid line) for tunneling into a homogeneous infinite Majorana chain as a function of $U = eV/\Gamma$ for $t = 0.1\Gamma$.

The Fano factor is defined as the ratio $F = S_{\rm shot}/2eI$. For the NS junction $F(V \to 0) = 2$ reflecting the transmission of cooper pairs through the superconductor. The Fano factor corresponding to the upper right panel of Fig. 1 (two MBSs) is plotted in Fig. 2. The zero -bias value of the Fano factor $F(V = 0) = 2$ does not depend on the value of $t \neq 0$ and is the same integer for four MBSs, the case in the lower left panel of Fig. 1. However, when the second Majorana state interacts with the normal lead, i.e., $\Gamma_{22} \neq 0$, $F(V = 0)$ is noninteger and decreasing, as shown by the solid line in Fig. 2. In contrast, with an odd number of MBSs $F(V \to 0) = 0$.

Considering next the case of an infinite homogeneous chain, where all near-neighbor couplings are identical, $t_{i,i+1} = t$, the Green's functions can be easily found.[16] The calculations of the shot noise are performed for $t/\Gamma = 0.1$ and compared with conductance in Fig. 3.

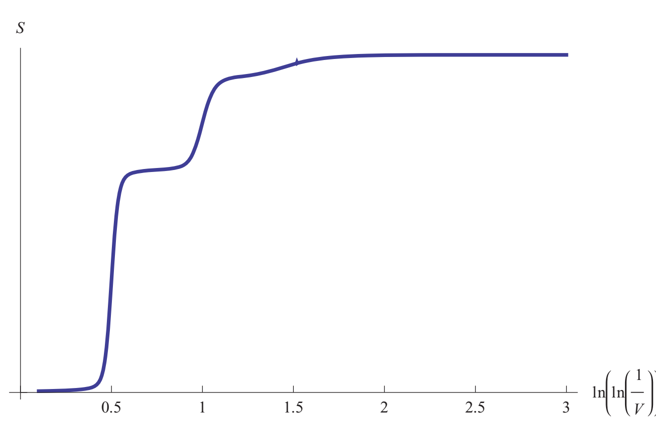

FIG. 4. (Color online) The shot noise power as a function of a double logarithmic voltage. The behavior of the noise power has similar sharp variations as the current (Ref. 15).

Disordered case: Isolated Majorana pairs. We consider next the case of a semi-infinite MBS chain with a random $t_{i,i+1}$ and only the first MBS coupled to the lead, $\Gamma_{ii} \sim \delta_{i,1}$. This system can be mapped to a system of independent MBS pairs[15] by using the analysis methods of the quantum Ising spin chain.[22,23] The key ingredient is to isolate the strongest bond at each decimation. The MBS pairs have an effective hopping $\epsilon_n^{\max} \approx e^{-e^{n/2}}$ (having a typical weight), leading to peaks of conductance at $eV = \epsilon_n$ with weights $\Gamma_n^2 \approx \exp[-8\Gamma_0 e^{n/2}]$; Γ_0 is a starting logarithmic flow parameter chosen here as $\Gamma_0 = 0.1$. For a given voltage the contribution to the noise is from pairs with $\epsilon_n^{\max} > eV$, resulting in jumps at $\epsilon_n = eV$. We estimate the averaged current and shot noise using our formulas (16), (17). The relevant frequencies $\omega \leqslant \epsilon_n^{\max}$ are selected by a Fermi function $f = 1/\{1 + \exp[(\omega - \epsilon_n)/\delta]\}$ with small δ and we approximate $G = [\omega - eV + i\delta']^{-1}$ with a small δ'. We perform calculations for $n \leqslant 6$. Figure 4 presents the shot noise as a function of a double-logarithmic voltage dependence (the voltage is measured in units of Γ). S_n and the average current J show similar behavior, although the jumps are of different heights.

Conclusion. We apply the standard Keldysh technique to calculate the shot noise power of a Majorana chain interacting with a normal lead. We find that the shot noise has markedly distinct forms for different numbers of MBSs and for long ordered or disordered chains. We show that for weak coupling to the lead the shot noise and the conductance are much stronger than in the corresponding SN junction. We calculate the Fano factor and show that its value crucially depends on whether even or odd numbers of MBSs are involved. The Fano factor is similar in magnitude to that in the SN case, in particular $F(V \to 0) = 2$ for even numbers of MBSs and if only one MBS is coupled to the lead; however, in the less likely case when more than one MBS is coupled to the lead (Fig. 2) $F(0)$ is noninteger.

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[1] L. Fu and C. L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
[2] M. Sato and S. Fujimoto, Phys. Rev. B **79**, 094504 (2009).
[3] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. **105**, 177002 (2010).
[4] J. Alicea, Phys. Rev. B **81**, 125318 (2010).
[5] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).
[6] A. C. Potter and P. A. Lee, Phys. Rev. Lett. **105**, 227003 (2010).
[7] X.-L. Qi and S.-C. Zhang, e-print arXiv:1008.2026.
[8] C. J. Bolech and Eugene Demler, Phys. Rev. Lett. **98**, 237002 (2007).
[9] J. Nilsson, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. Lett. **101**, 120403 (2008).
[10] C. Nayak, S. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[11] A. Y. Kitaev, Phys. Usp. **44**, 131 (2001).
[12] L. Fu and C. L. Kane, Phys. Rev. B **79**, 161408 (2009).
[13] K. T. Law, P. A. Lee, and T. K. Ng, Phys. Rev. Lett. **103**, 237001 (2009).
[14] J. Linder, Y. Tanaka, T. Yokoyama, A. Sudbø, and N. Nagaosa, Phys. Rev. Lett. **104**, 067001 (2010).
[15] V. Shivamoggi, G. Refael, and J. E. Moore, Phys. Rev. B **82**, 041405 (2010).
[16] K. Flensberg, Phys. Rev. B **82**, 180516(R) (2010).
[17] P. Roushan, J. Seo, C. V. Parker, Y. S. Hor, D. Hsieh, D. Qian, A. Richardella, M. Z. Hasan, R. J. Cava, and A. Yazdani, Nature (London) **260**, 1106 (2009).
[18] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, e-print arXiv:1006.4395.
[19] A. Kamenev and A. Levchenko, Adv. Phys. **58**, 197 (2009).
[20] L. Fu, Phys. Rev. Lett. **104**, 056402 (2010).
[21] K. Ya. M. Blanter and M. Büttiker, Phys. Rep. **336**, 1 (2000).
[22] N. E. Bonesteel and K. Yang, Phys. Rev. Lett. **99**, 140405 (2007).
[23] D. S. Fisher, Phys. Rev. B **51**, 6411 (1995).