# Aharonov-Bohm effect in the presence of dissipative environments

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We study a particle on a ring in presence of various dissipative environments. We develop and solve a variational scheme assuming low-frequency dominance. Our solution produces a renormalization-group (RG) transformation to all orders in the inverse dissipation strength and, in particular, reproduces known two-loop results. Our RG leads to a weak dissipation parameter, for which a weak-coupling expansion for the position correlation function shows a $1/\tau^2$ decay in imaginary time.

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## I. INTRODUCTION

The problem of interference and dephasing in presence of dissipative environments is of significance for a variety of experimental systems and a fundamental theoretical issue. The experimental systems include mesoscopic rings embedded on various surfaces where Aharonov-Bohm (AB) oscillations can be measured[1,2] and the related problem of decoherence at low temperatures.[3] A different type of experimental systems are cold atom traps created by atom chips.[4–6] The atom chip that produces a magnetic or electric trap for the cold atoms necessarily also produces noise. Our problem is then relevant for evaluating the interference amplitude of the cold atoms in presence of such noise.

As an efficient tool for monitoring the effect of the environment we follow a suggestion by Guinea[7] to find the AB oscillation amplitude as function of the radius $R$ of the ring; for free particles of mass $M$ this amplitude is the mean level spacing $\sim 1/MR^2$. Two types of environments were suggested to lead to an anomalous suppression, i.e., a stronger decrease in the oscillation amplitude than $1/R^2$: a Caldeira-Leggett (CL) bath as well as a charge-metal (CM) system, i.e., a charge on the ring interacting with a dirty metal environment. The CL system is of further interest since it can be mapped to the Coulomb-blockade problem[8,9] as well as to quantum dots at a distance from metallic gates.[10] The Coulomb box problem is of further recent interest in view of data on the quantization of the charge relaxation resistance[11,12] and related theoretical developments.[13–15]

The CL system has been extensively investigated by instanton methods,[16–18] by renormalization-group (RG) methods,[7,8] by a boundary field theory[19] and by Monte Carlo (MC) methods.[8,9,18,20] All methods show that the effective mass, defined as $B/R^2$, of the particle increases exponentially with the dissipation strength $\alpha$, i.e., $B\sim \alpha^{\mu}e^{\pi^2\alpha}$, with differences in the exponent $\mu$. In second-order RG $\mu=-1$ (Ref. 8) while instanton methods give either[16] $\mu=-2$ or[17] $\mu=-3$; the boundary field theory with MC gives $\mu=-2$. A variational approach[21] indicated a nonperturbative regime at strong $\alpha$. Since $\alpha=\gamma R^2$, where $\gamma$ is a friction coefficient, a length scale $\pi/\sqrt{\gamma}$ is identified;[7] this scale is a candidate for a dephasing length.

The CM system was investigated by RG methods[7] finding $B\sim R^{2+\mu'}$ with $\mu'\lesssim 1$ nonuniversal while MC data[22] shows $\mu'\approx 1.8$. Further MC simulations show that in fact $\mu'=0$, at least for weak coupling.[23] We study also a dipole-metal (DM) system, i.e., an electric dipole on a ring coupled to a dirty metal environment. This system can be realized by experiments on cold Rydberg atoms.[24]

In the present work, extending our previous report[25] we solve these systems by a variational method, assuming low-frequency dominance. We find that the variational method defines an RG scheme to all orders, reproducing a known RG equation[8] to two loops in the CL system. In the CM and DM systems, for either a charge or a dipole, we find that the effective mass remains $B/R^2\sim R^0$ for large $R$, as for free particles. Our RG leads to a weak-coupling dissipation parameter. The resulting action yields a weak-coupling expansion for the position correlation function, showing a $1/\tau^2$ decay in imaginary time. This decay is generic to all finite $R$ systems and indicates dephasing of an excited state. In the limit $R\to\infty$ the correlation probes degenerate states, however, the position correlation function does not decay in this limit, i.e., no dephasing.

In Sec. II we present the models. In Sec. III we define our variational method and show that the effective mass $B$ of the $m=0$ sector determines the curvature $\partial^2E_0/\partial\phi_x^2|_0$, where $E_0$ is the ground-state energy and $\phi_x$ is the flux through the ring; this curvature is a measure of the Aharonov-Bohm oscillation amplitude. In Sec. IV we simplify the variational equation by assuming low-frequency dominance or equivalently logarithmic dominance. In Sec. V we show that this method is equivalent to an RG scheme and, in particular, reproduces the known RG equation to second order in the CL system. In general the variational equation contains terms to all orders and is therefore expected to be superior to a second-order RG expansion. In Sec. VI we present explicit solutions for the CL and DM systems, as well as for a general case. Finally in Sec. VII we study the weak-coupling expansion showing a $1/\tau^2$ decay for the position correlation function.

## II. MODEL

In this section we derive the effective action in presence of a dissipative environment in terms of the angle $\theta_m(\tau)$, where $\tau$ is an imaginary time. The index $m$ specifies the winding number so that

$$\theta_m(\tau)=\theta(\tau)+2\pi m\tau/\beta, \tag{1}$$

where $\theta(0)=\theta(\beta)$ has periodic boundary condition and $\beta$ is the inverse temperature ($\beta\to\infty$ below). In presence of an external flux $\phi_x$ the partition sum has the form

$$Z=\sum_m \int \mathcal{D}\theta e^{2\pi i m\phi_x - S_1\{\theta_m\} - S_{int}\{\theta_m\}}. \quad (2)$$

As shown by Guinea,[7] the form of such an action in presence of a general dissipative bath the effective action can be written in terms of a kernel $K[\theta(\tau)-\theta(\tau')]$ that is periodic and allows in general a Fourier expansion

$$S_1\{\theta\}=\int_0^\beta d\tau \frac{MR^2}{2}\left(\frac{\partial\theta}{\partial\tau}\right)^2,$$

$$S_{int}\{\theta\}=\alpha\int_0^\beta\int_0^\beta d\tau d\tau' \frac{\pi^2T^2K[\theta(\tau)-\theta(\tau')]}{\sin^2[\pi T(\tau-\tau')]}$$

$$=\alpha\sum_n a_n \int_0^\beta\int_0^\beta d\tau d\tau' \frac{\pi^2T^2 \sin^2\{n[\theta(\tau)-\theta(\tau')]/2\}}{\sin^2[\pi T(\tau-\tau')]} \quad (3)$$

and $a_n$ depends on the type of bath. At $\tau\to\tau'$ (or at high frequencies $\omega$) one can expand the $\sin^2(\cdots)$ in Eq. (3) and then $S_{int}\to\frac{1}{4}\alpha\Sigma_n a_n n^2 \int d\omega|\omega||\theta_m(\omega)|^2$, identifying a dissipative system.

We consider now three types of environments and identify the coefficients $a_n$. First is the CL environment. It has harmonic oscillators coupled linearly to the particle's coordinate. The effective action is well known[26] for nonconfined coordinate $\mathbf{R}(\tau)$,

$$S_{int}^{\mathrm{CL}}=\gamma\int\int d\tau d\tau' \frac{\pi^2T^2[\mathbf{R}(\tau)-\mathbf{R}(\tau')]^2}{\sin^2[\pi T(\tau-\tau')]}, \quad (4)$$

where $\gamma$ is the dissipation parameter. When the particle is confined to a ring $\mathbf{R}(\tau)=\mathbf{R}[\cos\theta(\tau),\sin\theta(\tau)]$ the action becomes of the form of Eq. (3) with a single coefficient $a_1=1$ and $\alpha=\gamma R^2$.

Consider next the CM environment. It consists of a dirty metal that is characterized by its conductivity $\sigma$ and diffusion constant $D$. The particle on the ring has a charge $e$ and responds to the Coulomb potential of the metal $V[\mathbf{R}(\tau),\tau]$. The metal is assumed to be a Gaussian environment so that the interaction term (in imaginary time) of the partition sum can be averaged to obtain[22]

$$Z_{int}=\left\langle e^{-i\int_0^\beta V[\mathbf{R}(\tau),\tau]\mathbf{d}\tau}\right\rangle \equiv e^{-S_{int}} \quad (5)$$

and with $\int_q\equiv\int d^3q/(2\pi)^3$,

$$S_{int}=\frac{1}{2}e^2\langle V[\mathbf{R}(\tau),\tau]\mathbf{V}[\mathbf{R}(\tau'),\tau']\rangle$$

$$=\frac{1}{2}e^2T\sum_n\int_q \frac{4\pi}{q^2\epsilon(i|\omega_n|,q)}e^{i\mathbf{q}\cdot[\mathbf{R}(\tau)-\mathbf{R}(\tau')]-i\omega_\mathbf{n}(\tau-\tau')}, \quad (6)$$

where the propagator of the scalar potential[27] is given in terms of the dielectric function $\epsilon(i|\omega_n|,q)$ with $\omega_n=2\pi nT$ the Matsubara frequencies. At low frequencies and momenta $\epsilon(\omega,q)=1+\frac{4\pi\sigma}{-i\omega+Dq^2}$, valid at $q<1/\ell$, where $\ell$ is the mean-free path. Hence $1/\epsilon(i|\omega_n|,q)\approx(|\omega_n|+Dq^2)/4\pi\sigma$; the $Dq^2$ term yields an $\mathbf{R}(\tau)$ independent constant while

$$\sum_n|\omega_n|e^{-i\omega_n\tau}=\frac{-\pi T}{\sin^2[\pi T\tau]}, \quad (7)$$

hence with $k_F$ the Fermi wave vector and $r=R/\ell$ the charge coupled to a dirty metal has

$$\alpha=\frac{3}{8k_F^2\ell^2},$$

$$K(z)=1-[4r^2\sin^2(z/2)+1]^{-1/2}. \quad (8)$$

For $r\gg1$, $a_n\approx\frac{2}{\pi r}\ln(r/n)$ for $1<n\lesssim r$ and $a_n\approx0$ for $n\gtrsim r$. This model reduces to the CL one at $r\ll1$, where $a_1$ single $a_n$ survives.

A third realization of the action corresponds to the DM environment. Consider a particle with an electric dipole, whose direction is perpendicular to the ring, interacting with a metal. For the electric field $E_z=\partial_zV-\frac{1}{c}\partial_tA_z$, the $A_z$ propagator involves[27] $[\epsilon(i|\omega_n|,q)\omega_n^2+Dq^2]^{-1}$, which for $q\neq0$ can be expanded in $\omega_n^2$, hence it has no dissipative term $\sim|\omega_n|$; we keep then just the $\partial_zV$ term. The interaction with the fluctuating electric field $\mathbf{E}(\mathbf{r},\tau)$ is $p\int_0^\beta E_z[\mathbf{R}(\tau),\tau]d\tau$. A Gaussian average on the metallic environment then yields

$$S_{int}=\frac{1}{2}p^2\int_\tau\int_{\tau'}\langle\partial_zV[\mathbf{R}(\tau),\tau]\partial_\mathbf{z}\mathbf{V}[\mathbf{R}(\tau'),\tau']\rangle$$

$$=\frac{1}{2}p^2T\sum_n\int_q\frac{4\pi q_z^2}{q^2\epsilon(i|\omega_n|,q)}e^{i\mathbf{q}\cdot[\mathbf{R}(\tau)-\mathbf{R}(\tau')]-\mathbf{i}\omega_\mathbf{n}(\tau-\tau')}. \quad (9)$$

Therefore

$$\alpha=\frac{3}{8k_F^2\ell^2}\frac{p^2}{e^2\ell^2},$$

$$K(z)=1-\left(4r^2\sin^2\frac{z}{2}+1\right)^{-3/2}. \quad (10)$$

Hence, for large $r$, $a_n\sim\frac{1}{r}(1-\frac{n^2}{r^2})$ for $n\lesssim r$ and $a_n\approx0$ otherwise. Finally we note that a topological flux $\phi_x$ can be realized for an electric dipole.[28]

### III. VARIATIONAL METHOD

The partition sum can be rewritten by using the Poisson sum $\Sigma_m g(m)=\Sigma_K\int g(\phi)\exp(2\pi iK\phi)d\phi$ so that

$$Z=\sum_K\int_{-\infty}^{\infty}d\phi\int\mathcal{D}\theta e^{2\pi i\phi(K+\phi_x)-2\pi^2MR^2\phi^2/\beta-S_1\{\theta(\tau)\}-S_{int}\{\theta(\tau)+2\pi\phi\tau/\beta\}}$$

$$=\sum_K\int_{-\infty}^{\infty}d\phi e^{2\pi i\phi(K+\phi_x)-2\pi^2MR^2\phi^2/\beta}Z_\phi. \quad (11)$$

The variational method for $Z_\phi$ finds the best Gaussian approximation, i.e.,

$$S_0=\frac{1}{2\beta}\sum_{\omega_n}G^{-1}(\omega_n)|\theta(\omega_n)|^2 \quad (12)$$

so that the free energy $F$ in $Z_\phi=e^{-\beta F}$ has the variational form

$$\beta F_{var} = \beta F_0 + \langle S - S_0 \rangle_0 = \frac{1}{2\beta}\sum_{\omega_n} \{-\ln G(\omega_n) + [MR^2\omega_n^2 - G^{-1}(\omega_n)]G(\omega_n)\} + \langle S_{int}\rangle_0, \tag{13}$$

where $\langle \cdots \rangle_0$ is an average with respect to $\exp(-S_0)$ and $F_0$ is the free energy corresponding to $S_0$. Since

$$\begin{aligned} 2\langle \sin^2\{n[\theta(\tau) - \theta(\tau') + 2\pi\phi(\tau-\tau')/\beta]/2\}\rangle_0 &= 1 - \cos[2\pi n\phi(\tau-\tau')/\beta]\exp\{-n^2\langle[(\theta(\tau) - \theta(\tau')^2]\rangle_0/2\} \\ &= 1 - \cos[2\pi n\phi(\tau-\tau')/\beta]\exp\left\{-n^2T\sum_{\omega_n} G(\omega_n)[1-\cos(\omega_n\tau)]\right\}, \end{aligned} \tag{14}$$

the interaction term becomes

$$\langle S_{int}\rangle_0 = \beta\alpha\sum_n a_n \int_0^\beta \frac{d\tau}{2\tau^2}\left\{1 - \cos(2\pi n\phi\tau/\beta)e^{-n^2T\sum_\omega G(\omega)[1-\cos(\omega\tau)]}\right\}. \tag{15}$$

The variational equation $\delta F_{var}/\delta G(\omega_n)=0$ is then

$$G^{-1}(\omega) = MR^2\omega^2 + 2\alpha\sum_n a_n n^2\int_0^\beta d\tau \frac{1-\cos(\omega\tau)}{\tau^2}\cos(2\pi n\phi\tau/\beta)e^{-n^2\int(d\omega_1/2\pi)G(\omega_1)[1-\cos(\omega_1\tau)]}. \tag{16}$$

When the limit $\beta\to\infty$ is taken a cutoff $\omega_c$ may be introduced to control the short-time behavior so that the $\tau$ integral becomes $\int_{1/\omega_c}^\infty$. This cutoff represents a high-frequency limit of the bath degrees of freedom. Alternatively, the mass term serves also as a cutoff since it leads to convergence of the $d\omega_1$ integral in the exponent of Eq. (16).

In the following we will study the variational equation with $\phi=0$. To justify this, we show now that the effective mass $B$ of the $\phi=0$ system is indeed what is needed to find the Aharonov-Bohm oscillation amplitude at $T\to 0$. The effective mass is defined by $G^{-1}(\omega)=B\omega^2$ in the limit $\omega\to 0$ and is identified from Eq. (16) at $\beta\to\infty$ as

$$B = MR^2 + \frac{1}{2}\alpha\sum_n a_n n^2\int_0^\infty d\tau e^{-n^2\int(d\omega/2\pi)G(\omega)[1-\cos(\omega\tau)]}. \tag{17}$$

Form (11) implies that the fluctuations $\langle\phi^2\rangle\sim\beta$, hence the factor $\cos(2\pi n\phi\tau/\beta)\to 1+O(1/\beta)$ in Eq. (16) and the effective mass $B$ is $\phi$ independent. It is also necessary to check that the $\tau$ integrals converge: indeed at $\tau\to\infty$,

$$\int_0^\infty d\omega G(\omega)[1-\cos(\omega\tau)] \approx \tau^2\int_0^{1/\tau}\frac{d\omega}{2B} + \int_{1/\tau}^\infty d\omega G(\omega) \sim \tau/B,$$

hence a factor $e^{-n^2\tau/B}$ assures the convergence of the $\tau$ integrals.

The Aharonov-Bohm oscillation amplitude is usually measured[8,9] by the curvature of the free energy at $\phi_x=0$; since at $\phi=0$ we have $\partial G(\omega)/\partial\phi=0$ [from parity in $\phi$, see Eq. (16), and from analyticity in $\phi$] and $\partial F_{var}/\partial G=0$ (the variational condition) we obtain from Eqs. (13), (15), and (17),

$$\left.\frac{\partial^2\beta F_{var}}{\partial\phi^2}\right|_0 = \left.\frac{\partial^2\beta\langle S_{int}\rangle_0}{\partial\phi^2}\right|_0 = \frac{4\pi^2}{\beta}(B-MR^2). \tag{18}$$

The effect of $Z_\phi$ in the partition sum Eq. (11) is therefore to replace the factor $2\pi^2MR^2\phi^2/\beta$ by $2\pi^2B\phi^2/\beta$, i.e., the response to an external flux is that of a free particle with a mass renormalized to $B$. Higher-order terms produce only subdominant behavior in $1/\beta$, e.g., one expects a $\phi^4/\beta^3$ term. Our task is therefore to study the $\phi=0$ system and find this renormalized mass.

## IV. VARIATIONAL EQUATION

Before studying the full equation, it is instructive to study its perturbative regime. The lowest order is obtained by neglecting the exponent in Eq. (16), leading to

$$G^{-1}(\omega) = MR^2\omega^2 + \pi\omega\alpha\sum_n a_n n^2 \quad \omega \lesssim \omega_c. \tag{19}$$

This identifies the cutoff $\omega_c$ below which dissipative term dominates

$$\omega_c = \frac{\pi\alpha\sum_n a_n n^2}{MR^2}. \tag{20}$$

Consider next $\omega\ll\omega_c$ but still $\ln(\omega_c/\omega)\lesssim 1$. The next order in perturbation is obtained by using Eq. (19) in the exponent in Eq. (16) and expanding this exponent,

$$G^{-1}(\omega) = \pi\omega\alpha\sum_n a_n n^2\left[1 - \frac{n^2}{\pi^2\alpha\sum_m a_m m^2}\ln\frac{\omega_c}{\omega}\right]$$
$$= \pi\omega\alpha\sum_n a_n n^2\left[1 - \frac{1}{\alpha\kappa}\ln\frac{\omega_c}{\omega}\right] \quad \ln(\omega_c/\omega) \lesssim 1, \tag{21}$$

where the mass term is ignored for $\omega < \omega_c$ and $\kappa$ is a geometric parameter defined by

$$\kappa = \frac{\pi^2\left(\sum_n a_n n^2\right)^2}{\sum_n a_n n^4}. \tag{22}$$

The sums in Eq. (22) can be evaluated for each model from the second and fourth derivatives of $K(z)$ at $z=0$, leading to

$$\kappa = \pi^2 \quad \text{CL}$$
$$= \frac{2\pi^2 r^4}{r^2 + 9r^4} \quad \text{CM}$$
$$= \frac{6\pi^2 r^4}{r^2 + 15r^4} \quad \text{DM}. \tag{23}$$

A significant perturbative regime is possible for $\alpha\kappa \gg 1$. This strong dissipation condition can apply to the CL model if $R$ is large, though one needs to make sure that the CL model is still valid in that case. For the CM or DM models $\kappa$ is bounded by a number $\sim 1$ so that $\alpha \gg 1$ is needed. For usual dirty metals $k_F\ell \gtrsim 1$ so that for charge coupling with Eq. (8) the condition is not satisfied, unless the particle on the ring has a charge $e^* \gg e$. On the other hand, the dipole case may have a large $\alpha$ in Eq. (10) for large dipoles, e.g., in Rydberg atoms. In the following we use form (21) as a boundary condition for the full variational solution.

We proceed now to variational equation, that includes the significant range of $\omega \ll \omega_c$. It is convenient to study a derivative of Eq. (16),

$$\frac{d}{d\omega}G^{-1}(\omega) = 2\alpha\sum_n a_n n^2\int_0^\infty d\tau\frac{\sin(\omega\tau)}{\tau}e^{-n^2\int(d\omega_1/2\pi)G(\omega_1)[1-\cos(\omega_1\tau)]}. \tag{24}$$

The bare mass $M$ serves to define $\omega_c$ and then the $MR^2\omega^2$ term in Eq. (16) is neglected at $\omega < \omega_c$. If $\omega$ is sufficiently small then $\sin(\omega\tau)$ can be expanded leading to a $\sim\omega$ term. We therefore assume the form

$$G^{-1} = f(\omega) \quad \omega_0 < \omega < \omega_c,$$
$$G^{-1} = B\omega^2 \quad \omega < \omega_0. \tag{25}$$

The solution for $f(\omega)$ needs to satisfy boundary conditions, whose $\alpha$ dependence is determined by the perturbative expansion, Eq. (21),

$$f(\omega_c) = \pi\omega_c\alpha\sum_n a_n n^2,$$
$$f'(\omega_c) = \pi\alpha\sum_n a_n n^2 \cdot \eta(\alpha) \quad \eta(\alpha) = 1 + \frac{1}{\alpha\kappa} + O(1/\alpha^2),$$
$$f''(\omega_c) = \pi\sum_n a_n n^2 \cdot \frac{C(\alpha)}{\omega_c} \quad C(\alpha) = \frac{1}{\kappa} + O(1/\alpha). \tag{26}$$

We proceed to simplify Eq. (24). For $\omega > \omega_0$ the oscillating $\sin(\omega\tau)$ in Eq. (24) leads to a cutoff $\tau < 1/\eta_1\omega$, to be determined by matching to the perturbative regime. Hence

$$f'(\omega) = 2\omega\alpha\sum_n a_n n^2\int_0^{1/\eta_1\omega} d\tau e^{-n^2\int_0^{\omega_c}(d\omega_1/2\pi)G(\omega_1)[1-\cos(\omega_1\tau)]}. \tag{27}$$

The range $\int_{\omega_c}^\infty$ involves $G^{-1}(\omega)=MR^2\omega^2$ and contributes $\sim 1/(MR^2\omega_c)=[\pi\alpha\Sigma_n a_n n^2]^{-1}$ which is neglected for $\alpha \gg 1$. The $\tau$ integration is dominated by $\tau \approx 1/\eta_1\omega$, hence $1-\cos\omega_1\tau \approx 1-\cos(\omega_1/\eta_1\omega)$ is replaced by $\omega_1^2/2\omega^2\eta_1^2$ for $\omega_1 < \omega$ and by 1 for $\omega_1 > \omega$. This rough separation is to be justified by our main assumption that $\int_\omega^{\omega_c} d\omega_1/f(\omega_1)$ dominates this integral due to the low frequency decrease in $f(\omega_1)$. The terms from $\omega_0 < \omega_1 < \omega$, as well as those from $\omega_1 < \omega_0$, can be neglected if

$$\frac{1}{B\omega_0}, \quad \frac{1}{\omega^2}\int_{\omega_0}^{\omega}\frac{\omega_1^2 d\omega_1}{f(\omega_1)} \ll \int_\omega^{\omega_c}\frac{d\omega_1}{f(\omega_1)} \quad \text{condition(i)}. \tag{28}$$

Note that the second term on the left near $\omega_0$ is $\sim 1/B\omega_0$ while near $\omega_c$ it is $\sim 1/\alpha$ and negligible for large $\alpha$. We are interested in nonperturbative contributions, i.e., the range $\ln\omega_c/\omega \ll 1$ and, in particular, at $\omega=\omega_0$. In terms of $\omega_2 = 1/\tau$ we obtain

$$f'(\omega) = 2\omega\alpha\sum_n a_n n^2\int_{\eta_1\omega}^\infty\frac{d\omega_2}{\omega_2^2}e^{-n^2\int_{\eta_2\omega_2}^{\omega_c}d\omega_1/\pi f(\omega_1)}, \tag{29}$$

where as above, the precise location of the $\eta_2\omega$ cutoff should not be significant. The $\omega_2$ integration is dominated by its lower cutoff $\eta_1\omega$ so we expect that the exponent can be taken out of the integration with the replacement $\eta_2\omega_2 \to \eta_1\eta_2\omega$. More precisely, taking a derivative of Eq. (29) leads to

$$f'(\omega) = \pi\tilde{\eta}(\alpha)\alpha\sum_n a_n n^2 e^{-n^2\int_\omega^{\omega_c}d\omega_1/\pi f(\omega_1)} + \omega f''(\omega) \tag{30}$$

and $\eta_1 = 2/(\pi\tilde{\eta})$ and $\eta_1\eta_2 = 1$ are chosen. The coefficient $\tilde{\eta}(\alpha)$ is to be determined by the boundary condition (26).

To further simply the equation we assume now

$$f''(\omega) \ll \frac{f'(\omega)}{\omega} \quad \text{condition(ii)}, \tag{31}$$

leading to our main equation for $f(\omega)$,

$$f'(\omega) = \pi\eta(\alpha)\alpha\sum_n a_n n^2 e^{-n^2\int_\omega^{\omega_c} d\omega_1/\pi f(\omega_1)}. \tag{32}$$

The coefficient here is $\eta(\alpha)$, consistent with Eq. (26). Below we actually find that condition (ii) is not always satisfied and then we return to solve Eq. (30) instead of Eq. (32).

Finally, consider $\omega<\omega_0$. Equation (24) has then on the left $\frac{d}{d\omega}G^{-1}(\omega)=2B\omega$ while on the right it has the requested $\sim\omega$ form, except for a term where

$$I_1 = 2\omega\alpha\sum_n a_n n^2 \int_{1/\omega_0}^{1/\eta_1\omega} d\tau \exp\left\{-\frac{n^2}{\pi}\left[\int_0^{\omega_0}\frac{1-\cos\omega_1\tau}{B\omega_1^2}d\omega_1 + \int_{\omega_0}^{\omega_c}\frac{d\omega_1}{f(\omega_1)}\right]\right\}. \tag{33}$$

Since $\tau>1/\omega_0$ dominates, $\int_0^{\omega_0}\omega_1^{-2}(1-\cos\omega_1\tau)d\omega_1\approx\pi\tau/2$, hence

$$I_1 = 4\omega B\alpha\sum_n a_n\left(e^{-n^2/2B\omega_0} - e^{-n^2/2B\eta_1\omega}\right)e^{-n^2\int_{\omega_0}^{\omega_c}d\omega_1/\pi f(\omega_1)}. \tag{34}$$

The essential singularity in $\omega$ is negligible for $\omega<\omega_0$ when

$$B\omega_0 \gtrsim 1 \quad \text{condition(iii)}. \tag{35}$$

The remaining term at $\omega<\omega_0$ identifies $B$ and leads to a matching condition of the form $f'(\omega_0)=\eta' B\omega_0$. Continuity of derivatives yields $\eta'=2$, though we expect that the precise value of $\eta'$ will not be significant.

This completes the derivation of the equations for $B$ and $f(\omega)$. Equation (30) and (32) is to be solved with the boundary conditions in Eq. (26) [in case of Eq. (32) only the first two conditions are needed]. Furthermore, the matching conditions at $\omega_0$ are

$$f(\omega_0) = B\omega_0^2,$$

$$f'(\omega_0) = \eta' B\omega_0 \quad \eta' \approx 2. \tag{36}$$

## V. RG PROCEDURE

We present here an approximate solution of the variational equations by an RG method, which in some case (the CL case, see below) should be very close to exact. The idea is that an $\omega<\omega_c$ can serve as a new cutoff provided that the coupling $\alpha$ is renormalized into $\bar{\alpha}(\omega)$. The boundary condition (26) become therefore

$$f(\omega) = \pi\omega\bar{\alpha}(\omega)\sum_n a_n n^2,$$

$$f'(\omega) = \pi\bar{\alpha}(\omega)\sum_n a_n n^2\cdot\eta[\bar{\alpha}(\omega)],$$

$$f''(\omega) = \pi\sum_n a_n n^2\cdot\frac{C[\bar{\alpha}(\omega)]}{\omega}. \tag{37}$$

The number of needed equations depends on the order of the differential equation for $f(\omega)$, e.g., for Eq. (32) only the first two equations in Eq. (37) are needed. The functions $\eta(\alpha), C(\alpha)$ are known as an expansion in $1/\alpha$. As we find below, these functions can be determined explicitly by the variational equations.

Taking a derivative of the first equation in Eq. (37) yields a recursion relation for $\bar{\alpha}(\omega)$,

$$\omega\frac{d\bar{\alpha}(\omega)}{d\omega} = \bar{\alpha}(\omega)\{\eta[\bar{\alpha}(\omega)]-1\}. \tag{38}$$

Hence the boundary condition function $\eta(\alpha)$ determines the flow of the renormalized $\bar{\alpha}(\omega)$, i.e., it generates the RG flow to all orders for which $\eta(\alpha)$ is known. In particular, the flow terminates when $\eta(\alpha_c)=1$, i.e., $\alpha_c$ is a fixed point.

Before proceeding to solve for $\eta(\alpha)$, we show that the RG is equivalent to a solution of the form $f(\omega)=\omega g[K(\alpha)\omega]$, so that all the $\alpha$ dependence is included in the function $K(\alpha)$, i.e., the function $g(x)$ itself is $\alpha$ independent. This property is exact for our variational equation for the CL system (see below and Appendix B). For other systems the scaling function needs to be identified separately as, e.g., done in Sec. VI C for the CM system.

The first boundary condition from Eq. (26) is $g[K(\alpha)\omega_c]=\pi\alpha\Sigma_n a_n n^2$, hence $K'(\alpha)\omega_c g'[K(\alpha)\omega_c]=\pi\Sigma_n a_n n^2$. The second boundary condition is then

$$\begin{aligned} f'(\omega_c) &= g[K(\alpha)\omega_c] + K(\alpha)\omega_c g'[K(\alpha)\omega_c] \\ &= \left[\alpha + \frac{K(\alpha)}{K'(\alpha)}\right]\pi\sum_n a_n n^2 = \pi\alpha\eta(\alpha)\sum_n a_n n^2 \Rightarrow \eta(\alpha) \\ &= 1 + \frac{K(\alpha)}{\alpha K'(\alpha)}. \end{aligned} \tag{39}$$

In $g[K(\alpha)\omega]$ one can vary either $\alpha$ or $\omega$ with identical effects if $K(\alpha)\omega=K[\bar{\alpha}(\omega)]\omega_c$ (see also Appendix B) which by $d/d\omega$ yields

$$K'[\bar{\alpha}(\omega)]\frac{d\bar{\alpha}(\omega)}{d\omega}\omega_c = K(\alpha) = K[\bar{\alpha}(\omega)]\frac{\omega_c}{\omega} \tag{40}$$

and with Eq. (39) the flow, Eq. (38), is reproduced. We note also that the scaling functions $\eta(\alpha), C(\alpha)$, Eq. (37), can also be determined by rewriting the differential equation for $g(x)$ in terms of $y(g)=x(g)g'[x(g)]$ and its derivatives. This is possible under fairly general conditions, e.g., that $g(x)$ is monotonic and that the differential equation for $g(x)$ is homogenous [i.e., contains only $x^n g^{(n)}(x)$ terms].

So far the RG flow was determined, in general, without a necessity to identify the relevant differential equation. We proceed to show that the differential equation for $f(\omega)$ determines the function $\eta(\alpha)$ completely and therefore also the flow of $\bar{\alpha}(\omega)$. Consider Eq. (32) that leads to

$$f''(\omega_c) = \pi\eta\alpha\frac{\sum_n a_n n^4}{\pi f(\omega_c)} = \pi\eta\frac{\sum_n a_n n^2}{\kappa\omega_c}. \tag{41}$$

Differentiation of the second equation in Eq. (37) leads to

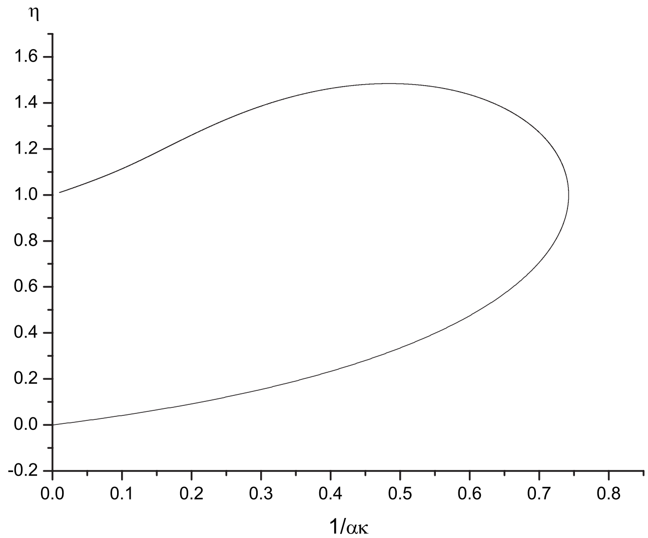


FIG. 1. Solution of Eq. (44) for the scaling function $\eta$ as function of $\kappa\alpha$. The branch at $\eta<1$ is not accessible with the initial values of Eq. (26). Note that $\kappa=\pi^2$ for the CL system.

$$f''(\omega)=\pi(\alpha\eta)'\frac{d\bar{\alpha}}{d\omega}\sum_n a_n n^2, \tag{42}$$

where $(\alpha\eta)'=\frac{d}{d\alpha}[\alpha\eta(\alpha)]$. Equating the last equation at $\omega_c$ with Eq. (41) leads, in terms of $\ell=-\ln\omega$, to

$$\left.\frac{d\alpha}{d\ell}\right|_{\omega_c}=-\frac{\eta}{\kappa(\alpha\eta)'}=-\frac{1}{\kappa}-\frac{1}{\kappa^2\alpha}+O(1/\alpha^2). \tag{43}$$

To obtain the expansion we use the perturbative form of $\eta$ in Eq. (26). Remarkably, result (43) is precisely the two-loop RG result For the CL system[8] (with $g=\pi^2\alpha/2$ in the notations or Ref. 8 and $\kappa=\pi^2$ for the CL system). Note that the same perturbative form in Eq. (38) yields only the first term $1/\kappa$.

Comparing Eqs. (38) and (43) yields

$$\eta=1+\frac{1}{\kappa\alpha\eta-1+\kappa\alpha^2\dfrac{d\eta}{d\alpha}}. \tag{44}$$

This relation generates a large $\alpha$ expansion with the leading form $\eta=1+(\kappa\alpha)^{-1}+O(\alpha)^{-2}$, consistent with the perturbation expansion Eq. (26). It is remarkable that the perturbation expansion allows for an asymptotic expansion of Eq. (44), i.e., a different form of $\eta(\alpha)$ in Eq. (26) would not allow such an expansion.

Figure 1 shows the solution of this equation with the exact analytic solution given in Appendix A. Note the turning point at $\eta=1$, $1/(\alpha\kappa)=0.742$. This corresponds to a fixed point at $\alpha_c$, i.e., if this point is reached at a frequency $\omega_a$ then at $\omega<\omega_a$ $\bar{\alpha}(\omega)=\alpha_c$ remains constant and $f(\omega)=\pi\alpha_c\omega$. This behavior is in fact inconsistent with the assumed form (24). Another difficulty is that continuity of $f'(\omega_0)/f(\omega_0)=\eta[\bar{\alpha}(\omega_0)]=\eta'$ needs $\eta[\bar{\alpha}(\omega_0)]\approx 2$, which is not achieved in Fig. 1.

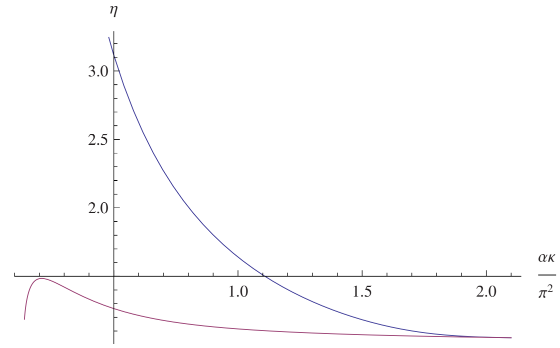


FIG. 2. (Color online) Solution of Eq. (46), upper line, for the scaling function $\eta$ as function of $\kappa\alpha$. The initial values for this solution are taken from a point on the solution of Eq. (44), lower line.

In the next section we evaluate $f(\omega)$ itself and show that the solution based on Eq. (32) does not satisfy criterion (ii) below some low frequency $\omega_b>\omega_a$. In the latter range one needs to address Eq. (30). We note that the $\omega f''(\omega)$ term in Eq. (30) is small at the initial range of $\omega$, e.g., at $\omega_c$ it is $O(1/\alpha)$ relative to the $f'(\omega)$ term. Therefore, to be consistent with the terms neglected due to the criteria (i), we need to start with Eq. (32), and only at the frequency $\sim\omega_b$ we shift to Eq. (30).

We proceed to study the RG form of Eq. (30). Taking a derivative of the second equation in Eq. (37) at $\omega=\omega_c$ yields

$$C(\alpha)=\alpha[\eta(\alpha)-1](\alpha\eta)'. \tag{45}$$

Equation (30) taken at $\omega_c$ yields $\tilde{\eta}(\alpha)=\eta(\alpha)-C(\alpha)/\alpha$. Next we evaluate $f'''(\omega_c)$ in two ways: first, by taking a derivative of Eq. (30) that leads to $f'''(\omega_c)=-\pi\tilde{\eta}(\alpha)\Sigma_n a_n n^2/(\kappa\omega_c^2)$. Second, by taking a derivative of the third equation in Eq. (37). Equating these two forms leads to, finally,

$$\eta(\alpha)-\frac{C(\alpha)}{\alpha}=\kappa\{C(\alpha)-C'(\alpha)\alpha[\eta(\alpha)-1]\}. \tag{46}$$

Together with Eq. (45) this is a second-order differential equation for $\eta(\alpha)$. We solve this equation by matching at some $\alpha$ to the solution of Eq. (44), as shown in Fig. 2. Curiously, Eq. (46) has an exact solution $\eta=1+1/(\kappa\alpha)$ which gives the one-loop solution in Eq. (43). As mentioned above, we apply Eq. (30) only below some low frequency $\sim\omega_0$, to be studied in the next section i.e., $\eta=1+\frac{1}{\kappa\alpha}$ does not have then the proper boundary conditions.

## VI. SOLUTIONS FOR VARIOUS SYSTEMS

We present now explicit solutions for $f(\omega)$ and study the validity criteria. We start with the mathematically simplest case, the CL system.

### A. Caldeira-Leggett system

Considering Eq. (32) we obtain by differentiating

$$f''(\omega)=\frac{f'(\omega)}{\pi f(\omega)}, \tag{47}$$

which upon integration yields

$$f'(\omega) = \pi^{-1} \ln[Kf(\omega)], \tag{48}$$

where $K$ is an integration constant. Further integration yields

$$\mathrm{li}[Kf(\omega)] = \pi^{-1}K(\omega - \omega_a) + \mathrm{li}[Kf(\omega_a)], \tag{49}$$

where at $\omega_a$, $\bar{\alpha}(\omega)$ reaches the fixed point of Eq. (44), i.e., $\bar{\alpha}(\omega_a)=\alpha_c$, anticipating that this equation is not valid all the way to $\omega_a$; here $\mathrm{li}(x)=\int_0^x \frac{dx'}{\ln x'}$ is the log integral function. $K$ is determined by the $\omega_c$ values

$$f'(\omega_c) = \eta\pi\alpha = \pi^{-1} \ln[Kf(\omega_c)] \tag{50}$$

so that

$$Kf(\omega_c) = e^{\eta\pi^2\alpha} \gg 1 \tag{51}$$

and from $f(\omega_c)=\pi\alpha\omega_c$ we have

$$K = \frac{e^{\pi^2\alpha\eta}}{\pi\alpha\omega_c}. \tag{52}$$

Equation (48) at $\omega=\omega_a$ yields

$$Kf(\omega_a) = e^{\eta'\pi B\omega_a}. \tag{53}$$

For $B$ and $\omega_a$ we need to solve the coupled equations

$$\int_{e^{\eta'\pi B\omega_a}}^{e^{\eta\pi^2\alpha}} \frac{dz}{\ln z} = \frac{K}{\pi}(\omega_c - \omega_a),$$

$$KB\omega_a^2 = e^{\eta'\pi B\omega_a}. \tag{54}$$

An explicit solution requires an asymptotic expansion of $\mathrm{li}(x)$, which is provided by our RG method. As discussed in Sec. V, the solution has the form $f(\omega)=\omega g(K\omega)$ such that $g(x)$ is $\alpha$ independent. Equations (40) and (52) then yield $f(\omega)=\pi\omega\bar{\alpha}(\omega)$, where $\bar{\alpha}(\omega)$ the solution of

$$K\omega = \frac{e^{\pi^2\bar{\alpha}(\omega)\eta[\bar{\alpha}(\omega)]}}{\pi\bar{\alpha}(\omega)}. \tag{55}$$

Inverting this relation we find

$$f(\omega) = \pi\omega\bar{\alpha}(\omega) = \frac{\omega}{\pi\eta}\ln[\pi\bar{\alpha}(\omega)K\omega] = \frac{\omega}{\pi\eta}\ln\left\{\frac{K\omega}{\eta\pi}\ln[\pi\bar{\alpha}(\omega)K\omega]\right\} = \frac{\omega}{\pi\eta}\ln\left[\frac{K\omega}{\eta\pi}\ln\left(\frac{K\omega}{\eta\pi}\ldots\right)\right] \tag{56}$$

and at least two ln embeddings are needed for a large $\alpha$ solution, i.e.,

$$f(\omega) = \frac{\omega}{\pi}\ln\left[\frac{K\omega}{\pi}\ln\frac{K\omega}{\pi}\right] + O\left[\frac{\omega}{\ln(K\omega)}\right]. \tag{57}$$

The boundary condition at $\omega_a$ is $K\omega_a=e^{\eta'\pi B\omega_a}/B\omega_a$ so that $g(K\omega_a)=B\omega_a$ becomes

$$g\left(\frac{e^{\eta'\pi B\omega_a}}{B\omega_a}\right) = B\omega_a. \tag{58}$$

This equation does not involve the large parameter $\alpha$, hence $B\omega_a\approx 1$, $K\omega_a\approx 1$, and the effective mass at scale $\omega_a$ is

$$B \approx \frac{1}{\omega_a} \approx \frac{e^{\pi^2\alpha}}{\alpha\omega_c}. \tag{59}$$

We note that Eq. (55) implies that $\bar{\alpha}(\omega_a)=O(1)$, i.e., in the vicinity of the fixed point $\alpha_c=0.14$.

We check now the conditions (i)–(iii) for $\omega$ near $\omega_a$. For condition (i) we use Eq. (32) at $\omega=\omega_a$, where $f'(\omega_a)=\pi\eta[\bar{\alpha}(\omega_a)]\bar{\alpha}(\omega_a)=O(1)$, hence

$$\int_{\omega_a}^{\omega_c} \frac{d\omega_1}{\pi f(\omega_1)} \approx \ln\alpha. \tag{60}$$

Hence the condition (i) is satisfied only for $\ln\alpha\gg 1$. The condition (ii) corresponds to $\omega f''(\omega)=\omega f'(\omega)/\pi f(\omega)\ll f'(\omega)$, hence $\pi f(\omega)/\omega=\pi^2\bar{\alpha}(\omega)\gg 1$. At $\omega_a$ this condition fails.

We consider therefore the previous solution as valid only down to a frequency $\omega_b$ to be determined below. At $\omega<\omega_b$ we use the more complete Eq. (30). As seen in Fig. 2, below $\omega_b$ the slope $\eta(\alpha)$ increases rapidly toward the value $\eta'\approx 2$ which determines $\omega_0$. A numerical fit to Fig. 2 and use of Eq. (38) yields a weak $\alpha$ dependence, i.e., $\omega_0\approx\omega_b/\alpha^x$ with $x\approx 0.2$. Neglecting this effect, we identify $\omega_0\approx\omega_b$ and check the various conditions. Consider first

$$\int_{\omega_b}^{\omega_c} \frac{d\omega_1}{f(\omega_1)} = -\pi\ln\frac{f'(\omega_b)}{\pi\alpha} = \pi\ln\frac{\alpha}{\bar{\alpha}(\omega_b)\eta[\bar{\alpha}(\omega_b)]}.$$

Since $B\omega_0\approx f(\omega_b)/\omega_b=\pi\bar{\alpha}(\omega_b)$, condition (i) is satisfied for any choice of $\omega_b$ such that $\bar{\alpha}(\omega_b)\gg 1$, e.g., $\bar{\alpha}(\omega_b)=\alpha^\nu$ with $\nu<1$,

$$\omega_0 \approx \frac{\alpha\omega_c}{\bar{\alpha}(\omega_b)}e^{-\pi^2\alpha+\pi^2\bar{\alpha}(\omega_b)}$$

with $\eta,\eta[\bar{\alpha}(\omega_b)]\approx 1$; since $\nu<1$ $\bar{\alpha}(\omega_b)<\alpha$ and $\omega_0\ll\omega_c$ and provides a huge range where Eq. (32) is valid.

Consider next condition (ii),

$$\frac{\omega_b f''(\omega_b)}{f'(\omega_b)} = \frac{\omega_b}{\pi f(\omega_b)} = \frac{1}{\pi\bar{\alpha}(\omega_b)} \ll 1,$$

which is also satisfied when $\bar{\alpha}(\omega_b)\gg 1$; condition (iii) is also obvious from $B\omega_0\approx\pi\bar{\alpha}(\omega_b)$. Finally we find

$$B \approx \frac{\pi\alpha^{2\nu}}{\alpha\omega_c}e^{\pi^2\alpha-\pi^2\alpha^\nu}. \tag{61}$$

The choice of the exponent $\nu$ is a balance for allowing a maximal range for Eq. (32), which neglects the terms in the three conditions on equal footing, and the necessity of satisfying the conditions. We expect then $\nu\ll 1$. We note that result (61) is closer to the Monte Carlo form[20] $B\sim e^{\pi^2\alpha}/\alpha^2\omega_c$ than Eq. (59) above.

### B. Study of the general case

We present here an analysis of the general case using the asymptotic expansion in the parameter $\kappa\alpha$. Define the function

$$F(x) = \pi\eta\alpha\sum_n a_n n^2 e^{-n^2 x/\pi} \tag{62}$$

so that Eq. (32) becomes

$$f'(\omega) = F\left[\int_\omega^{\omega_c} d\omega_1/f(\omega_1)\right]. \tag{63}$$

The boundary condition for this first-order equation is $f(\omega_c)=\pi\omega_c\Sigma_n\alpha_n n^2$ while the condition $f'(\omega_c)=F(0)$ follows from the equation itself. We now generate a second-order equation

$$\frac{d}{d\omega}F^{-1}[f'(\omega)] = \frac{-1}{f(\omega)}. \tag{64}$$

Multiplying by $f'(\omega)$ and integrating yields

$$-H[f'(\omega)] + H[f'(\omega_c)] = \int_\omega^{\omega_c} f'(\omega_1)\frac{d}{d\omega_1}F^{-1}[f'(\omega_1)]d\omega_1$$

$$= -\ln\frac{f(\omega_c)}{f(\omega)}. \tag{65}$$

Hence in term of the function $H(y)=\int^{F^{-1}(y)}F(x)dx$, determined up to one integration constant, one obtains

$$H[f'(\omega)] = -\ln[Kf(\omega)],$$

$$K = \frac{e^{-H[f'(\omega_c)]}}{f(\omega_c)}. \tag{66}$$

For the Caldeira-Leggett system one can choose $H[f'(\omega)]=-\pi f'(\omega)$, i.e., a $\alpha$-independent function, which leads to the solution in the previous section. In the general case however the function $H(y)$ depends explicitly on $\alpha$, in the form $H(y)=\alpha h(y/\alpha)$, where $h$ is the reciprocal function of $F/\alpha$. Hence it is not strictly possible to look for a solution of the form $f(\omega)=\omega g(K\omega)$ with a $\alpha$ independent $g$. Explicit integration of Eq. (66) is then required with proper matching, Eq. (25), at frequency $\omega_0$ but this will not be attempted here in full generality. For $H(y)$ a power law however, one can redefine a scaling function as shown in the next section.

Instead we will follow an approximate method which is consistent with the one-loop RG (see also Appendix C). The idea is to determine the integration constant $K$ by an expansion near $\omega_c$ where

$$y = F(x) = \pi\eta\alpha\sum_n a_n n^2 - \eta\alpha x\sum_n a_n n^4.$$

This identifies $F^{-1}(y)$ and the function $H$ is then, to first order in $f'(\omega)-f'(\omega_c)$,

$$H[f'(\omega)] = -\frac{f'(\omega_c)f'(\omega)}{\eta\alpha\sum_n a_n n^4}.$$

Using the boundary condition and Eq. (66),

$$K = \frac{e^{\eta\alpha\kappa}}{\alpha\pi\omega_c\sum_n a_n n^2}. \tag{67}$$

We can now rederive the RG Eq. (44) by a solution of the form $f(\omega)=\omega g(K\omega)$ such that $g(x)$ does not depend explicitly on $\alpha$. At $x=K\omega_c$ we have

$$g(x) = \pi\alpha\sum_n a_n n^2,$$

$$xg'(x) = \frac{K(\alpha)}{K'(\alpha)}\pi\sum_n a_n n^2 \tag{68}$$

so that $f'(\omega_c)=\eta\pi\alpha\Sigma_n a_n n^2$ becomes Eq. (44).

The reasoning below Eq. (55) can now be repeated so that $f(\omega)$ is generated by repeated ln embeddings. For the effective mass $B$ we need the boundary condition at $\omega_0$, i.e., $H[f'(\omega_0)]=H(\eta' B\omega_0)=-\ln(KB\omega_0^2)$ and $g(K\omega_0)=B\omega_0$ which yield an equation for the product $B\omega_0$,

$$g\left(\frac{e^{-H(\eta' B\omega_0)}}{B\omega_0}\right) = B\omega_0. \tag{69}$$

The relation $g(K\omega_0)=B\omega_0$ determines then $\omega_0$ and hence, finally, $B$. Before reaching $\omega_0$, at $\sim\alpha\omega_0$, we expect the modification as discussed in the CL system (previous section), leading to a change in the exponent $\mu$.

### C. Charge-metal system

In this system we define a mean-free path $l$, Fermi wave vector $k_F$, and then[7,22] the Fourier expansion is identified by

$$\left[1 - \frac{1}{\sqrt{4r^2\sin^2(z/2)+1}}\right] = \sum_n a_n(1-\cos nz)/2. \tag{70}$$

Hence $a_n \approx \frac{2}{\pi r}\ln(r/n)$ for $1<n\lesssim r$, where $r=R/l$ while $a_n\approx 0$ otherwise. Applying $d^2/dz^2$ and $d^4/dz^4$ at $z=0$ we get $\Sigma_n a_n n^2=2r^2$ and $\Sigma_n a_n n^4=2r^2+18r^4$.

We show first that the dependence of the effective mass on the radius is $B\sim R^2$ when $R\to\infty$, as for a free particle. We rely here on the proof of Sec. III, within the variational method, that the effective mass can be found from $\phi=0$ in Eq. (11). The action has then the form

$$S_{int}\{\theta_0\} = \sum_{n=1}^{n=r}(1/r)\ln(r/n)\bar{S}\{n\theta_0(\tau)\} \to -\int_0^1 dx\ln x\bar{S}\{x\bar{\theta}_0(\tau)\},$$

where $\bar{S}$ is a functional of $\theta_0(\tau)$, the latter is rescaled as $\bar{\theta}_0(\tau)=r\theta_0(\tau)$. The action (including the free term $S_1$) is then $r$ independent and therefore the effective mass for $[\bar{\theta}_0(\tau)]^2$ is $r$ independent, which after unscaling yields $B\sim r^2$.

We proceed to study the variational solution at large $r$ and large $\alpha$. While realistic metals have $\alpha\lesssim 1$, this study supplements MC studies,[23] done at small $\alpha$. In the region $x \ll \pi$ and in the large $r$ limit, the function $F(x)$ takes the form,

$$F(x) = r^2 \tilde{F}(r^2 x), \quad \tilde{F}(y) \approx 2\pi\eta\alpha \int_0^1 dz \ln(1/z) z^2 e^{-z^2 y/\pi} \tag{71}$$

with $\tilde{F}(0) = 2\pi\eta\alpha$ and $\tilde{F}(y) \approx 2\pi^{5/2}\eta\alpha y^{-3/2}$ for large $y = r^2 x$. For even larger values of $x > \pi$, i.e., $y > r^2\pi$ the function behaves as in the CL regime $F(x) = \frac{2\pi\eta\alpha e}{r} e^{-x/\pi}$.

It is useful to define the rescaled function via $f(\omega) = r^2 \tilde{f}(\omega)$ so that in regime $x < \pi$ the variational equation becomes, for $\omega < \omega_c$,

$$\tilde{f}'(\omega) = \tilde{F}\left[\int_\omega^{\omega_c} \frac{d\omega_1}{\tilde{f}(\omega_1)}\right], \tag{72}$$

which is now independent of $r$. It can be solved, in principle, and assuming that the matching frequency $\omega_0$ occurs in this region $x < \pi$ we get $B = r^2 \tilde{B}$ with $r$-independent conditions $\tilde{f}(\omega_0) = \tilde{B}\omega_0^2$ and $\tilde{f}'(\omega_0) = \eta' \tilde{B}\omega_0$, for $\omega_0$ and $\tilde{B}$. Hence $\omega_0$ and $\tilde{B}$ are $r$ independent in the large $r$ limit (they depend on $\alpha$) and we recover that $B \sim r^2$.

Note that in the regime of large $y$ we can use the asymptotic form and the Eq. (66) can then be integrated as

$$H[\tilde{f}'(\omega)] = -2\pi(2\pi\eta\alpha)^{2/3}[\tilde{f}'(\omega)]^{1/3} = -\ln[K\tilde{f}(\omega)], \tag{73}$$

where $K$ is an $\alpha$-dependent integration constant. We note that a scaling function can be defined via

$$\bar{f}(\omega) = (2\pi\eta\alpha)^2 \tilde{f}(\omega) = \omega g(\bar{K}\omega),$$

where $\bar{K} = K/(2\pi\eta\alpha)^2$. $g(x)$ satisfies $\pi[g(x) + xg'(x)] = \ln^3[xg(x)]$, hence $g(x)$ is $\alpha$ independent, except through its argument $\bar{K}$.

If Eq. (73) is used to identify the matching point, Eq. (36), at $\omega_a$ then

$$K\tilde{B}\omega_a^2 = e^{\eta' \tilde{B}\omega_a (2\pi\eta\alpha)^{2/3}},$$

$$g\left[\frac{1}{(2\pi\eta\alpha)^2 \tilde{B}\omega_a} e^{\eta' \tilde{B}\omega_a (2\pi\eta\alpha)^{2/3}}\right] = (2\pi\eta\alpha)^2 \tilde{B}\omega_a \tag{74}$$

so that

$$(2\pi\eta\alpha)^2 \tilde{B}\omega_a = O(1). \tag{75}$$

As we show momentarily, this analysis fails near $\omega_a$ as conditions (i) and (ii) fail. As in the CL case, we define $\omega_b$ ($\omega_a < \omega_b \ll \omega_c$) so that below $\omega_b$ the corrected Eq. (30) is applied and then we expect $\omega_0 \approx \omega_b$.

Alternatively, we can use a scaling form for $\bar{f}(\omega)$ as in Eq. (37), i.e., $\bar{f}(\omega) = 2\pi\omega\bar{\alpha}(\omega)$, $\bar{f}'(\omega) = 2\pi\omega\bar{\alpha}(\omega)\eta[\bar{\alpha}(\omega)]$. Hence Eq. (73) becomes

$$\bar{K}\omega = \frac{1}{\pi\bar{\alpha}(\omega)} e^{\{\pi^2 \bar{\alpha}(\omega)\eta[\bar{\alpha}(\omega)]\}^{1/3}}, \tag{76}$$

which, at $\omega = \omega_c$, identifies $K = e^{2\pi^2\eta\alpha}/(2\pi\alpha\omega_c)$. Matching at $\omega_0 \approx \omega_b$ and using Eq. (72),

$$\frac{1}{\tilde{B}\omega_0} \approx \frac{(2\pi\eta\alpha)^2}{2\pi\bar{\alpha}(\omega_b)}. \tag{77}$$

Note that replacing $\omega_b \to \omega_a$ recovers Eq. (75), confirming that $\bar{\alpha}(\omega_a) = O(1)$.

For condition (i) we need

$$\int_{\omega_b}^{\omega_c} \frac{d\omega_1}{\tilde{f}(\omega_1)} = \frac{\pi(2\pi\eta\alpha)^2}{\{2\pi\bar{\alpha}(\omega)\eta[\bar{\alpha}(\omega)]\}^{2/3}} \gg \frac{1}{\tilde{B}\omega_0}, \tag{78}$$

which is satisfied if $\bar{\alpha}(\omega_b) \gg 1$; clearly at $\omega_a$ this condition fails. For condition (ii), by a derivative of Eq. (72), we obtain

$$\frac{\omega_b f''(\omega_b)}{f'(\omega_b)} = \frac{3}{2} \frac{\{\pi\bar{\alpha}(\omega_b)\eta[\bar{\alpha}(\omega_b)]\}^{2/3}}{2\pi^2\bar{\alpha}(\omega_b)} \sim [\bar{\alpha}(\omega_b)]^{-1/2} \ll 1,$$

which is also satisfied when $\bar{\alpha}(\omega) \ll 1$. Finally, condition (iii) is satisfied since $B \sim r^2$.

To obtain the effective mass $B$, Eq. (76) leads to

$$\omega_b = \frac{\alpha\omega_c(2\pi\eta\alpha)^2}{\bar{\alpha}(\omega_b)} e^{-2\pi^2\eta\alpha + [2\pi^4\bar{\alpha}(\omega_b)]^{1/3}},$$

where $K$ from Eq. (69) is used and $\kappa = 2\pi^2/9$ at $r \gg 1$. As in the CL case, we choose $\bar{\alpha}(\omega) \sim \alpha^\nu$ with $\nu \ll 1$ so that $\omega_b \ll \omega_c$, providing a large integration regime for Eq. (32). Finally, the effective mass is

$$\tilde{B} = \frac{2\pi\bar{\alpha}^2(\omega_b)}{(2\pi\eta\alpha)^4\alpha\omega_c} e^{2\pi^2\eta\alpha - [\pi^2\bar{\alpha}(\omega_b)]^{1/3}} \approx \frac{1}{(2\pi)^3\alpha^5\omega_c} e^{2\pi^2\alpha}. \tag{79}$$

It is easy to see that the condition that the frequency $\omega_0$ belongs to the scale-invariant regime and not in the CL regime is $x \ll \pi$, i.e.,

$$\pi r^2 \gg \int_{\omega_0}^{\omega_c} \frac{d\omega}{\tilde{f}(\omega)}, \tag{80}$$

the right-hand side (rhs) which can be determined from the solution, depends only on $\alpha$ and not on $r$, hence this sets a minimum radius as a condition.

## VII. CORRELATION FUNCTION

### A. Small $\alpha$ perturbation theory

Independently of the variational method it is also useful to consider the straight small $\alpha$ perturbation theory of the action, Eq. (11). We consider first the effect of the $\phi$ integration in Eq. (11). Perturbation expansion in $\alpha$ leads in general to a $\phi$ dependence of the form $e^{2\pi i\phi\tilde{\tau}/\beta}$, where $\tilde{\tau}$ is a linear combination of the various time variables $\tau_i$ in the expansion. The $\phi$ integral is then

$$\sum_K \int_{-\infty}^{\infty} d\phi e^{2\pi i\phi(K+\phi_x)-2\pi^2 MR^2\phi^2/\beta+2\pi i\phi\tilde{\tau}/\beta} = \sum_K e^{-\beta(K+\phi_x+\tilde{\tau}/\beta)^2/(2MR^2)}. \tag{81}$$

We expect that the various $\tau_i$ integrations converge so that at $\beta\to\infty$ the limit $\tilde{\tau}/\beta \ll K+\phi_x$ can be taken and then the sum is dominated by $K=0$ when $|\phi_x|<\frac{1}{2}$. We show this explicitly for the first order below.

For $\phi_x=0$ and $\beta=\infty$ of Eq. (11) we can therefore consider $Z_{\phi=0}$. Here we compute the correlation function of cos $\theta$ order by order in $\alpha$. The zeroth order is obtained from the free-particle action $S_1$, Eq. (3), and given by

$$\langle\cos[\theta(\tau)-\theta(0)]\rangle_0 = \exp\left\{-\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}[1-\cos(\omega\tau)]\frac{1}{MR^2\omega^2}\right\} = \exp\left(-\frac{1}{2}\omega_M|\tau|\right), \tag{82}$$

where we have defined $\omega_M=\frac{1}{MR^2}$. To perform the expansion we take the $\beta\to\infty$ limit in the time integrals since these are found to be convergent while we keep $\beta$ in the $\phi$ dependence. For $\phi_x=0$ we will rewrite the interaction in Eq. (11),

$$S_{int} = -\frac{1}{2}\alpha\sum_{n\geq 1} a_n \int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} d\tau d\tau' \frac{\cos\{n[\theta(\tau)-\theta(\tau')+2\pi\phi(\tau_1-\tau_2)/\beta]\}-1}{|\tau-\tau'|^2}. \tag{83}$$

The first-order correction is obtained from the connected average using Eq. (81),

$$\langle\cos(\theta(\tau)-\theta(0))\rangle_1 = \frac{\alpha}{2}\sum_{n\geq 1} a_n \int_{\tau_1}\int_{\tau_2} S_K(\tau,\tau_1-\tau_2)\frac{1}{(\tau_1-\tau_2)^2}\langle e^{i[\theta(\tau)-\theta(0)+n\theta(\tau_1)-n\theta(\tau_2)]}\rangle_{0,c}$$
$$= \frac{\alpha}{2}a_1\int_{\tau_1}\int_{\tau_2} S_K(\tau,\tau_1-\tau_2)\frac{e^{-\omega_M/2(|\tau|+|\tau_1-\tau_2|)}}{(\tau_1-\tau_2)^2}(e^{\omega_M/2(|\tau-\tau_1|+|\tau_2|-|\tau_1|-|\tau-\tau_2|)}-1),$$

where $S_K(\tau,x)=\sum_K e^{-\beta[K+(\tau+x)/\beta]^2/(2MR^2)}$. As we find the integrals are indeed convergent so that $\beta\to\infty$ can be taken and $S_K(\tau,x)\to 1$. We have discarded exponentially decaying terms in $\tau$ such as produced by $n>1$. It is important to note that the starting integral is convergent for $\tau_1\approx\tau_2$. To see that one can symmetrize in $\tau_1,\tau_2$ the term in parenthesis: expansion for $\tau_1\approx\tau_2$ then yields an additional $(\tau_1-\tau_2)^2$ term. This is a general property for all connected averages: the small time apparent singularity is absent. Indeed, expanding the cosine in the vertex, Eq. (83), and contracting the two fields with times external to the vertex yields $\nabla\theta(\tau_1)\theta(\tau)$ correlations, which are always bounded in the action $S_1$.

Although integral (D3) is tedious to compute, its large $\tau$ behavior is easily extracted. It is clear that, assuming a Wick decoupling for the $e^{i\theta}$ factors, then the exponential decay of each two-point correlator fixes the value of $\theta_1\to 0$ and of $\theta_2\to\tau$, leading to a $1/\tau^2$ form. To be more precise, the mass term forces the variables $\tau_1=0+O(1/\omega_M)$ and $\tau_2=\tau+O(1/\omega_M)$, i.e., this is the region which dominates the integral in Eq. (D3) at large $\tau\gg 1/\omega_M$. Integration is then easy in that region and amounts to replace $\exp(-\frac{1}{2}\omega_M|\tau_1|)\to\frac{4}{\omega_M}\delta(\tau_1)$ and $\exp(-\frac{1}{2}\omega_M|\tau_2-\tau|)\to\frac{4}{\omega_M}\delta(\tau_2-\tau)$. Note that since the result is only a function of $\omega_M\tau$ the large $\tau$ limit is the same as the large $\omega_M$ limit at fixed $\tau$. This yields

$$\langle\cos[\theta(\tau)-\theta(0)]\rangle_1 \approx \alpha a_1\frac{8}{\omega_M^2\tau^2} \tag{84}$$

at large $\tau$. The amplitude is confirmed by the detailed calculation of the integral given in Appendix D, as well as by a numerical check.

We note that this $1/\tau^2$ decay is in general agreement with the constraints derived in Ref. 30 for the long-range XY model, very similar to our CL model I. There it was shown that for strictly ferromagnetic long—range interactions the spin correlation cannot decay slower than the interaction. For the DM model, the $1/\tau^2$ has a $a_1\sim 1/r$ coefficient, hence it vanishes in the $r\to\infty$ limit.

The second-order correction can be written as

$$\langle\cos[\theta(\tau)-\theta(0)]\rangle_2 = \frac{\alpha^2}{8}\sum_{n\geq 1} a_n\sum_{n'\geq 1} a_{n'}\int_{\tau_1}\cdots\int_{\tau_4}\frac{\langle e^{i[\theta(\tau)-\theta(0)+n\theta(\tau_1)-n\theta(\tau_2)+n'\theta(\tau_3)-n'\theta(\tau_4)]}\rangle_c}{(\tau_1-\tau_2)^2(\tau_3-\tau_4)^2}. \tag{85}$$

At large $\tau$ the main contribution comes from $\tau_2\approx\tau$, $\tau_3\approx 0$, and $\tau_1\approx\tau_4$ (and one deduced by exchange 1, 2 with 3, 4) and yields

$$\langle\cos[\theta(\tau)-\theta(0)]\rangle_2 \approx 8\alpha^2 a_1^2\frac{1}{\omega_M^3}\int_{\tau_1}\frac{1}{(\tau_1-\tau)^2\tau_1^2}. \tag{86}$$

This integral looks divergent at small times but it is meant to be regularized for $\tau_1$ near zero by the region $\tau_2 \approx \tau$, $\tau_3 \approx \tau_1 \approx \tau_4 \approx 0$ in the above integral (85). This mainly replaces the $1/\tau_1^2$ factor by a $\sim\langle\cos[\theta(\tau_1)-\theta(0)]\rangle_1$ factor regular at $\tau_1 = 0$. Similarly, the singularity at $\tau_1 = \tau$ is smoothed by proper integration of Eq. (85) in the region $\tau_2 \approx \tau_1 \approx \tau_4 \approx \tau$, $\tau_3 \approx 0$. Since it is regularized at small times on times of order $\sim 1/\omega_M$ the above integral behaves as

$$\langle\cos[\theta(\tau)-\theta(0)]\rangle_2 \approx 8\alpha^2 a_1^2 \frac{A}{\omega_M^2 \tau^2}. \qquad (87)$$

To compute the coefficient $A$ we need to perform carefully the integrals in the small time regularization region. The question of universality of this amplitude is discussed in Appendix E. We will not attempt that here but simply note that there is no large time divergence in the above integral, i.e., the coefficient $A$ is finite and does not contain any log divergence.

### B. Large $\alpha$ behavior via matching

Let us now estimate the correlation function in the large $\alpha$ limit and we restrict to CL for simplicity. For $\tau$ not too large we can just use the straight large $\alpha$ perturbation theory,

$$\langle\cos[\theta(\tau)-\theta(0)]\rangle = \exp\left\{-\int_{-\omega_c}^{\omega_c} \frac{d\omega}{2\pi}[1-\cos(\omega\tau)]\frac{1}{\pi\alpha|\omega|}\right\}$$
$$\approx \frac{1}{(\omega_c\tau)^{1/\pi^2\alpha}}, \qquad (88)$$

which presumably is valid only for $\ln(\omega_c\tau) \ll \pi^2\alpha$. For larger time one needs to consider renormalization of the dissipation. We use the analysis of the previous sections. For large $\alpha$ we expect that we can use the fixed point action which is of form (3) with renormalized parameters, i.e., near $\alpha_c = 1/(0.742\kappa)$ and with the mass $M$ replaced by the renormalized mass $B/R^2$ with $B=B(\alpha)$. To get an estimate of the correlation function at large $\tau$, we can now use the above result (84) for the small coupling expansion replacing $\alpha$ by $\alpha^*$ and $\omega_M$ by $1/B(\alpha)$ [according to Eq. (61)]. Since $\alpha^*$ is not strictly small this will only provide an estimate. One gets, keeping the dominant exponential term,

$$\langle\cos[\theta(\tau)-\theta(0)]\rangle \sim \frac{e^{2\pi^2\alpha}}{(\omega_c\tau)^2}, \qquad (89)$$

which we expect to be valid for $\ln(\omega_c\tau) \gg \pi^2\alpha + O(\ln\alpha)$. Equation (89) matches Eq. (88) at $\ln\omega_c\tau \approx \pi^2\alpha$.

## VIII. DISCUSSION

We have studied two types of environments: (i) the CL system with relevance to small rings, $r<1$, or to the Coulomb box problem and (ii) the dirty metal environment, that can couple to either a charge (CM system) or an electric dipole (DM system), with relevance to experiments on cold Rydberg atoms.[24]

For the CL system, the variational method was shown to be equivalent to an RG scheme, reproducing the known two-loop result.[8] Our method provides an expansion to all orders in $1/\alpha$ and leads to the renormalized mass Eq. (61), which is close to the result of the boundary field theory[19] and the MC data.[20]

At small $\alpha$ we find a regular expansion without ln divergences up to second order, i.e., the RG $\beta$ function for $\alpha$ seems to vanish perturbatively. Since we know that the RG flow of $\alpha$ at large $\alpha$ is toward smaller values of $\alpha$, there seems to be three main possible scenarios: (i) the flow toward small $\alpha$ becomes much slower, either exponentially due to some putative nonperturbative corrections, or to some higher order in $\alpha$, (ii) there is a line of fixed points for $\alpha < \alpha_c$ with some termination point $\alpha_c$, and (iii) there is a infinite set of fixed points at small alpha with accumulation at zero, or (iv) a one parameter RG is consistent only above a fixed point.[29]

In fact the result $\langle\cos[\theta(\tau)]\cos[\theta(0)]\rangle \sim 1/\tau^2$ is a robust one, relating to a theorem on an XY model on a lattice.[30] This result is derived in first order in $\alpha$ is remarkable. For large $\alpha$ one should use the scaling to small $\alpha$ and then use the former result.

For the dirty metal problem we show that at large $r$ the whole action scales with $r^2$, leading an $r$-independent effective mass $B/R^2$. Furthermore, we find a scaling form for large $r$ and large $\alpha$ that leads to the renormalized mass, Eq. (79). For the CM system $\alpha \lesssim 1$ from Eq. (8), yet for the DM system a large $\alpha$ may be realized in Eq. (10) if the dipole has $p > e\ell$, i.e., the extension of the Rydberg atom needs to be $>\ell$. The large $\alpha$ solution is useful also as a complement to the small $\alpha$ MC data [23] at $\alpha=0.19$, showing saturation of the effective mass with $r$. Therefore, the claims for an $r$-dependent mass[22] are in contrast with both weak and strong $\alpha$ results. We note also that the result to first order $\langle\cos[\theta(\tau)]\cos[\theta(0)]\rangle \sim a_1/\tau^2$ vanishes as $a_1 \sim \frac{1}{r} \to 0$ suggesting that the nonlinearities associated with $\alpha$ become weaker in the large $r$ limit. We believe that the correspondence of the variational method with the scaling forms is a useful and instructive guide for studies of large variety of nonlinear systems.

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## APPENDIX A: THE $\eta$ parameter

We solve here Eq. (44) for $\eta(\alpha)$. We change variable to $x=1/\alpha$ and then to $y(x)=\eta(x)/x$,

$$\eta'(x) = -\frac{\eta(x)}{\kappa[\eta(x)-1]} + \frac{\eta(x)}{x}, \qquad (A1)$$

$$y'(x) = -\frac{y(x)}{\kappa[xy(x)-1]}, \tag{A2}$$

therefore,

$$-\kappa\frac{xy(x)-1}{y(x)} = \frac{1}{y'(x)} = x'(y), \tag{A3}$$

$$x'(y) + \kappa x(y) = \frac{\kappa}{y}. \tag{A4}$$

A general solution of the homogenous part is $x = C_1 e^{-\kappa y}$ while for a solution to the full equation substitute $x = A(y)e^{-\kappa y}$ so that $A'(y) = \kappa e^{\kappa y}/y$, hence

$$A(y) = \kappa\int \frac{e^{\kappa y}}{y}dy = \kappa\,\mathrm{Ei}(\kappa y) + C_2, \tag{A5}$$

$$x(y) = (C_1 + C_2)e^{-\kappa y} + \kappa\,\mathrm{Ei}(\kappa y)e^{-\kappa y}, \tag{A6}$$

where Ei is the exponential integral function, with the asymptotic expansion

$$\mathrm{Ei}(z) = \frac{e^z}{z}\left[1 + \frac{1}{z} + \frac{2!}{z^2} + \cdots\right] \quad z \to \infty. \tag{A7}$$

The boundary condition gives at $x \to 0$,

$$x = \frac{1}{y} + \frac{1}{\kappa y^2} + Ce^{-\kappa y} + \cdots, \tag{A8}$$

$$y = \frac{1}{x} + \frac{1}{\kappa} + \frac{C}{x^2}e^{-\kappa/x} + \cdots, \tag{A9}$$

$$\eta(x) = xy(x) = 1 + \frac{x}{\kappa} + \frac{C}{x}e^{-\kappa/x}, \tag{A10}$$

where $C = C_1 + C_2$. Comparison with Eq. (6) shows that $C = 0$, a remarkable result. The solution is then

$$1 = \kappa\alpha\,\mathrm{Ei}(\kappa\alpha\eta)e^{-\kappa\alpha\eta}. \tag{A11}$$

This result is plotted in Fig. 1 for $\eta$ as function of $1/(\kappa\alpha)$.

## APPENDIX B: THE LOG INTEGRAL

We present here the mathematical result, i.e., solving the log integral by using RG and deriving an asymptotic solution. Consider the equation

$$\tilde{f}'(y) = \frac{1}{\pi}\ln[\tilde{K}\tilde{f}(y)] \tag{B1}$$

with the boundary condition $\tilde{f}(1) = \pi\alpha$. This is the equation for the CL system, Eq. (48), in the text, defining $f(\omega) = \omega_c\tilde{f}(\omega/\omega_c)$. The constant $\tilde{K}$ is parametrized as

$$\tilde{K} = e^{\pi^2\alpha\eta}/\pi\alpha$$

so that $\tilde{f}'(1) = \pi\alpha\eta$. In general, we expect that any pair $(\alpha,\eta)$ will produce a solution $\tilde{f}(y)$.

We rewrite the solution in the form

$$\tilde{f}(y) = yg\left(\frac{e^{\pi^2\alpha\eta}}{\pi\alpha}y\right) \tag{B2}$$

so that the boundary condition at $y=1$ is

$$g[\tilde{K}(\alpha)] = \pi\alpha. \tag{B3}$$

Imagine now a varying boundary condition $\alpha$ and that the function $g$ does not depend explicitly on $\alpha$; this implies that $\eta(\alpha)$ must be chosen in a specific way. Equation (B1) can be written as

$$g(x) + xg'(x) = \frac{1}{\pi}\ln[xg(x)], \tag{B4}$$

where $x = \tilde{K}y$. Taking a derivative of Eq. (B3) yields $(d\tilde{K}/d\alpha)g'(\tilde{K}) = \pi$ so that Eq. (B4) at $x = \tilde{K}$ yields

$$\alpha + \frac{\tilde{K}(\alpha)}{\tilde{K}'(\alpha)} = \frac{1}{\pi^2}\ln[\tilde{K}(\alpha)\pi\alpha] = \alpha\eta(\alpha) \tag{B5}$$

leading to a differential equation for $\eta(\alpha)$,

$$\eta = 1 + \frac{\tilde{K}(\alpha)}{\alpha\tilde{K}'(\alpha)} = 1 + \frac{1}{\pi^2\alpha\eta - 1 + \pi^2\alpha^2\dfrac{d\eta}{d\alpha}}. \tag{B6}$$

Integrating this equation from any initial values $(\alpha,\eta)$ yields a function $\eta(\alpha)$; the full solution at $y<1$ is then obtained as $\tilde{f}(y) = \pi y\bar{\alpha}(y)$, where the function $\bar{\alpha}(y)$ is determined by the choice $\tilde{K}[\bar{\alpha}(y)] = \tilde{K}(\alpha)y$. Indeed one then has

$$f(y) = yg(\tilde{K}y) = yg\{\tilde{K}[\bar{\alpha}(y)]\} = \pi y\bar{\alpha}(y).$$

Therefore one needs to invert the algebraic relation $\tilde{K}[\bar{\alpha}(y)] = \tilde{K}(\alpha)y$ to find $\tilde{f}(y)$, leading to a form like Eq. (56). In general, however, Eq. (B6) is not easier than the original Eq. (B1), except for the initial values $(\alpha=\infty,\ \eta=1)$ which allow an asymptotic expansion with large $\alpha$. That our physical system satisfies this special tuning is most remarkable.

## APPENDIX C: INTEGRATION OF RG

It is instructive to study the one-loop RG of the dirty metal system and compare with the variational solution. Consider then the RG equations,[7] which can also be read off from Eq. (21),

$$\frac{d\alpha_n}{d\ell} = -\frac{n^2\alpha_n}{\pi^2\sum_n \alpha_n n^2} \tag{C1}$$

with defined $\alpha_n = \alpha a_n$. Let us consider the function $F(x)$ of Eq. (62) setting $\eta=1$ there. It becomes now $\ell$ dependent with

$$\frac{dF_\ell(x)}{d\ell} = \frac{F'_\ell(x)}{F_\ell(0)}. \tag{C2}$$

Change to the variable

$$F_\ell(0)\frac{\partial}{\partial \ell}=\frac{\partial}{\partial \mu} \tag{C3}$$

so that

$$\left[\frac{\partial}{\partial \mu}-\frac{\partial}{\partial x}\right]F_\ell(x)=0, \tag{C4}$$

which has the solution

$$F_{\ell(\mu)}(x)=F_{\ell(0)}[x+\mu(\ell)] \tag{C5}$$

and with $\mu=\int_0^\ell d\ell/\pi F_\ell(0)$ we have the general solution

$$F_\ell(0)=F_0\left[\int_0^\ell \frac{d\ell'}{F_{\ell'}(0)}\right]. \tag{C6}$$

Note that it has some formal similarity to the variational Eq. (63) if we define $\ell=-\ln \omega/\omega_c$. We proceed to study the dirty metal case with $F_\ell=F_\ell(0)$,

$$F_\ell=\frac{2\pi^{5/2}\alpha}{r}\left[\int_0^\ell \frac{d\ell'}{F_{\ell'}}\right]^{-3/2}. \tag{C7}$$

Hence by differentiating

$$-\frac{2}{3}\left(\frac{2\pi^{5/2}\alpha}{r}\right)^{2/3}\frac{\partial F_\ell/\partial\ell}{F_\ell^{2/3}}=1. \tag{C8}$$

Therefore,

$$2\left(\frac{2\pi^{5/2}\alpha}{r}\right)^{2/3}[F_{\ell=0}^{1/3}-F_\ell^{1/3}]=\ell=\ln\left(\frac{\omega_c}{\omega_c^R}\right), \tag{C9}$$

where $\omega_c^R$ is a renormalized cutoff. RG terminates at $F_\ell=1 \ll F_0=2\pi\alpha r^2$, hence

$$2\left(\frac{2\pi^{5/2}\alpha}{r}\right)^{2/3}(2\pi\alpha r^2)^{1/3}=\ln\left(\frac{\omega_c}{\omega_c^R}\right). \tag{C10}$$

Since powers of $r$ cancel $\omega_c\sim\omega_c^R$, i.e., the frequency at which the RG is stopped is independent of $r$, a conclusion also obtained in the text.

## APPENDIX D: CALCULATION OF AN INTEGRAL

The integral given in the text, upon rescaling $\omega_M\tau\to\tau$ is computed as

$$\begin{aligned}
&\frac{1}{2}\int_{\tau_1}\int_{\tau_2}\frac{e^{-1/2(|\tau|+|\tau_1-\tau_2|)}}{(\tau_1-\tau_2)^2}[e^{\omega_M/2(|\tau-\tau_1|+|\tau_2|-|\tau_1|-|\tau-\tau_2|)}-1]\\
&=\frac{1}{2}e^{-|\tau|/2}\Bigg\{\int_{-\infty}^{-\tau/2}dx\frac{e^x}{x^2}[(1-x-\tau/2)(e^\tau-1)-\tau]\\
&+\int_{-\tau/2}^{0}dx\frac{e^x}{x^2}[e^{-2x}(1+\tau/2+x)+x-\tau/2-1]\\
&+\int_0^{\tau/2}dx\frac{e^{-x}}{x^2}[(1-e^{-2x})(1-\tau/2+x)-2x]+\int_{\tau/2}^{\infty}dx\frac{e^{-x}}{x^2}[(1-e^{-\tau})(1-x+\tau/2)-\tau]\Bigg\}\\
&=\frac{1}{4}e^{-2\tau}\left(-e^{3\tau/2}(3\tau-4)\mathrm{Ei}\left(-\frac{3\tau}{2}\right)-e^{\tau/2}(e^{2\tau}\tau+\tau+4)\mathrm{Ei}\left(-\frac{\tau}{2}\right)+e^{3\tau/2}\left\{\tau\left[\mathrm{Ei}\left(\frac{\tau}{2}\right)+\log(27)\right]+8-\log(81)\right\}-4e^{2\tau}-4\right)
\end{aligned} \tag{D1}$$

$$=\frac{1}{4}\tau^2[-2\log(\tau)-2\gamma+1+\log(4)]+O(t^3),\quad \tau\ll 1 \tag{D2}$$

$$=\frac{8}{\tau^2}+\frac{384}{\tau^4}+O(\tau e^{-\tau/2}),\quad \tau\gg 1. \tag{D3}$$

## APPENDIX E: STRUCTURE OF HIGHER ORDERS IN SMALL $\alpha$ perturbation

The discussion of the first- and second-order corrections in the text suggested that the large time behavior of the integrals could be obtained from a Wick theorem on the $e^{i\theta}$ fields with a $\delta$-function correlator, e.g., the structure of the second-order correction, Eq. (85), at large time is an integral dominated by $\tau_2\approx\tau$, $\tau_3\approx 0$, and $\tau_1\approx\tau_4$, i.e., by the region such that the charges in $e^{in\theta(\tau_i)}$ should compensate. This was found to generically lead to $1/\tau^2$ decay. A question is then whether the amplitude of this decay can be obtained to all orders using this property, and how does it depend on the details of the short time cutoff.

Let us first consider a toy model with the same property. It is a Gaussian theory of partition sum $\int Dz(\tau)e^{-S}$ (for $a_n = \delta_{n1}$),

$$S = \frac{\omega_M}{4}\int_\tau z(\tau)z^*(\tau) - \frac{1}{2}\alpha\int_{\tau,\tau'} z(\tau)g(\tau-\tau')z^*(\tau'), \quad \text{(E1)}$$

where $z(\tau)$ plays the role of $e^{i\theta(\tau)}$ and $g(\tau)\approx 1/\tau^2$ at large $\tau$. Being a Gaussian variable it reproduces, for $\alpha=0$ the propagator $\langle z(\tau)z^*(\tau')\rangle_0=4\omega_M^{-1}\delta(\tau-\tau')$. There is thus some similarities in the perturbation expansion in $\alpha$ with the original model. Here however it is immediate to obtain

$$\langle z_\tau z^*_{\tau'}\rangle = G(\tau-\tau'), \quad G(\omega) = \frac{4}{\omega_M - 2\alpha g(\omega)}. \quad \text{(E2)}$$

Let us consider two examples for the short-time cutoff function in Eq. (E1). (i) $g(\omega)=\pi\omega_M e^{-|\omega|/\omega_M}$ corresponds to a Lorentzian $g(\tau)=1/(\omega_M^{-2}+\tau^2)$. The coefficient of $G(\tau)\sim A/\tau^2$ at large $\tau$ is obtained from the expansion $G(\omega)=G(0)-\pi A|\omega|+\cdots$ and reads $A=8\alpha/[(1-2\alpha\pi)\omega_M^2]$ by expanding Eq. (E2), (ii) $g(\omega)=-\pi|\omega|e^{-\omega^2/\omega_M^2}$ which gives $A=8\alpha/\omega_M^2$, i.e., only a first-order contribution, all higher orders being zero.

The above example shows that the amplitude $A$ can depend on the short-time cutoff beyond leading order. In that case it was however easily calculable by convolutions. To check whether one can indeed predict the $1/\tau^2$ coefficient more generally, let us consider now the following general discrete XY model of partition sum,

$$Z = \prod_i \int_0^{2\pi}\frac{d\theta_i}{2\pi}\exp\left(\sum_{k,l} g_{kl}e^{i(\theta_k-\theta_l)}\right) \quad \text{(E3)}$$

setting $g_{kk}=0$ for convenience. A calculation using MATHEMATICA then gives, for $x\neq 0$ (here $x,p,q$ belong to an arbitrary lattice),

$$\langle e^{i(\theta_x-\theta_0)}\rangle = g_{0x} + \sum_p g_{0p}g_{px} + \sum_{p,q\neq 0,x} g_{0p}g_{pq}g_{qx} - \frac{1}{2}g_{x0}g_{0x}^2 \quad \text{(E4)}$$

$$+ \sum_{p\neq q\neq r\neq p\neq 0,x} g_{xp}g_{pq}g_{qr}g_{rx} - \sum_{p\neq 0,x} g_{x0}g_{0x}g_{0p}g_{px} - \frac{1}{2}\sum_{p\neq 0,x}(g_{0p}g_{xp}g_{px}^2 + g_{p0}g_{px}g_{0p}^2 + g_{0x}^2 g_{xp}g_{0p}) + O(g^5). \quad \text{(E5)}$$

Taking as an example a one-dimensional chain with discrete $\tau$ values, and $g_{\tau\tau'}=g(\tau-\tau')\sim 1/(\tau-\tau')^2$ at large $\tau-\tau'$, one sees that up to order $O(g^3)$ (included) the structure captured by model, Eq. (E1), is correct to predict the coefficient of the $1/\tau^2$ decay of $\langle e^{i(\theta_\tau-\theta_0)}\rangle$. Indeed, up to that order, the $1/\tau^2$ decay can be obtained from the convolution of the $g$ kernel. However some new terms arise at order $g^4$ which are not of the above form and do contribute to the $1/\tau^2$ decay and the calculation of $A$ becomes more complicated than in model, Eq. (E1).


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