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Dephasing of a particle in a dissipative environment

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Abstract

The motion of a particle in a ring of length L is influenced by a dirty metal environment whose fluctuations are characterized by a short correlation distance $\ell \ll L$. We analyze the induced decoherence process, and compare the results with those obtained in the opposing Caldeira–Leggett limit ($\ell \gg L$). A proper definition of the dephasing factor that does not depend on a vague semiclassical picture is employed. Some recent Monte Carlo results about the effect of finite temperatures on 'mass renormalization' in this system are illuminated.

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1. Introduction

What is dephasing of a particle that has an energy E if it is subject to a fluctuating environment that has a temperature T? In particular what is dephasing close to equilibrium ($E \sim T$), and what happens in the limit $T \to 0$? This question has fascinated the mesoscopic community during the last two decades [1–7]. Our purpose is to study this question within the framework of linear response theory for a general characterization of the environment. In the Caldeira–Leggett (CL) framework [8, 9][1], the effect of the environment is characterized by a friction coefficient η and by a temperature T. But, more generally [10–12], it has been emphasized that the proper way to characterize the environment is by its form factor $\tilde{S}(q, \omega)$. The form factor contains information on both the temporal and the spatial aspects of the fluctuations, and in particular one can extract from it not only T and η, but also the spatial correlations. Typically (but not always) these spatial correlations can be characterized by a correlation distance ℓ.

So now we ask the refined question: given $\tilde{S}(q, \omega)$, what is dephasing? But first we have to say what do we mean by dephasing. In [10, 11] the CL approach has been generalized. Namely, it has been realized that an environment with a given $\tilde{S}(q, \omega)$ can be modeled as a set of harmonic oscillators. Then it is possible to apply the Feynman–Vernon formalism in order to trace them out. Using a semiclassical point of view, the propagator is expressed

[1] We emphasize here the *Ohmic* CL model, which is of relevance in the present context as a limiting case for a dirty metal environment. Obviously in general one may consider non-Ohmic models, where memory kernels are involved while the notion of a friction constant η becomes ill defined.

as a sum over pairs of classical trajectories. One observes that after time t, the interference contribution (from the off-diagonal terms in the double sum) is suppressed by a factor P_φ. This factor is interpreted as a dephasing factor, and its expression can be cast into the form $P_\varphi = \exp(-F(t))$ with

$$F(t) = \int \mathrm{d}\boldsymbol{q} \int \frac{\mathrm{d}\omega}{2\pi} \tilde{S}(\boldsymbol{q}, \omega) \tilde{P}(-\boldsymbol{q}, -\omega; t) t, \tag{1}$$

where the $\mathrm{d}\boldsymbol{q}$ integration measure depends on the dimensionality. In the semiclassical treatment, $\tilde{S}(q, \omega)$ would be the *symmetrized* form factor of the environment and $\tilde{P}(q, \omega; t)$ would be the classical *symmetric* power spectrum of the motion. There are some subtleties in the definition of $\tilde{P}(q, \omega; t)$ that we are going to discuss later on. In particular, we note that $\tilde{P}(q, \omega; t)$ may have weak dependence on t because a finite time interval is considered.

It has been further argued in [12] that due to inherent limitations of the semiclassical (stationary phase) approximation, the physically correct procedure is to use the *non*-symmetrized quantum versions of $\tilde{S}(q, \omega)$ and $\tilde{P}(q, \omega; t)$. This point has been further discussed in [13]. However, a proper derivation of equation (1), that does not rely on the semiclassical framework, has not been introduced. One objective of the present paper is to extend the derivation of the above formula beyond the semiclassical context.

It is important to realize that equation (1) is capable of reproducing all the established results about dephasing in normal metals, including the high temperature $\propto T$ dependence of the dephasing time, the low temperature $\propto T^{3/2}$ dependence in the case of a diffusive particle and the $\propto T^2$ dependence in the ballistic regime. At finite temperatures there is a finite time scale $\hbar/T$ that allows the approximation $F(t) \approx \Gamma_\varphi t$, and hence the notion of the dephasing rate Γ_φ is well defined. In the limit $T \to 0$, we always have $\Gamma_\varphi \to 0$. This, however, does not exclude the sub-exponential (power law) decay of P_φ.

Indeed, it is well known [6] that for a Brownian particle with CL environment the function $F(t)$ grows as $\log(t)$ at zero temperature, implying sub-exponential dephasing at $T = 0$. It is a common misconception that equation (1) with *non*-symmetrized spectral functions fails to reproduce this $\log(t)$ spreading. We shall dwell on this point later on in this paper. Furthermore, we shall study whether similar sub-diffusive behavior can be found for general $\tilde{S}(q, \omega)$.

During the last decade, the study of a particle in a ring has become a paradigm for the study of ground state anomalies. [14–18]. Besides being a prototype problem that possibly can be realized as a mesoscopic electronic device, it is also of relevance to experiments with particles that are trapped above an 'atom chip' device [19–21], where noise is induced by nearby metal surfaces. Significant progress has been achieved in analyzing the equilibrium properties of this prototype system, in particular the dependence of the ground state energy on the Aharonov Bohm flux through the ring. The derivations of the dephasing factor using equation (1) for the ring problem is a major objective of the present paper. In this context, there is a growing understanding that the study of dephasing is intimately connected with the study of mass renormalization at low temperatures. We believe that our results shed new light on some recent findings [22] that have been obtained using Monte Carlo data for the temperature dependence of the mass-renormalization effect.

The outline of this paper is as follows. In sections 2 and 3 we characterize the environment by the power spectrum of its fluctuations, and then in sections 4–6 we derive the formula for the dephasing factor. This formula is applied in sections 7 and 8 to the calculation of the dephasing of a particle of mass M in a ring of length L. The results depend crucially on the correlation distance ℓ of the fluctuating environment. They shed light on some new findings regarding mass renormalization in this system as explained in section 9. It is conjectured that

the mass-renormalization effect involves a measure for coherence. Some further discussion of the theoretical framework is presented in section 10.

2. The characterization of a fluctuating field

We regard the environment as a fluctuating field $\mathcal{U}(x,t)$. See appendix A for more details on its Hamiltonian modeling. The fluctuations of the environment are characterized by a form factor:

$$\tilde{S}(q,\omega) = \int\int [\langle \hat{\mathcal{U}}(x,t)\hat{\mathcal{U}}(0,0)\rangle]\, \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}qx}\, \mathrm{d}t\, \mathrm{d}x, \tag{2}$$

where the expectation value assumes that the bath is in a stationary state of its unperturbed Hamiltonian. The force operator is formally defined as $\mathcal{F} = -\mathcal{U}'(x,t)$, where the derivative is taken with respect to x. The force–force correlation function is obtained via double differentiation of the correlation function. In particular, the local power spectrum of the fluctuating force is

$$\tilde{S}(\omega) = \int \frac{\mathrm{d}q}{2\pi} q^2 \tilde{S}(q,\omega). \tag{3}$$

An Ohmic environment is characterized by

$$\tilde{S}_{\mathrm{ohmic}}(\omega) = \frac{2\hbar\eta\omega}{1-\mathrm{e}^{-\hbar\omega/T}} \qquad \text{for} \quad |\omega| < \omega_c. \tag{4}$$

The friction coefficient characterizes the response of the environment to an x variation ('force proportional to velocity'). Accordingly, it is given by the Kubo formula:

$$\eta = \lim_{\omega\to 0} \frac{1}{2\hbar\omega}[\tilde{S}(\omega) - \tilde{S}(-\omega)] = \frac{\tilde{S}(\omega=0)}{2T}. \tag{5}$$

For a strictly Ohmic bath, the friction coefficient is frequency independent and the first equality holds for any $\omega < \omega_c$ (no need to take a limit). The generalization of the above to three dimensions is straightforward. The position coordinate becomes $\boldsymbol{x} = (x,y,z)$ and accordingly q should be replaced by $\boldsymbol{q} = (q_x, q_y, q_z)$, with integration measure $\mathrm{d}^3\boldsymbol{q}/(2\pi)^3$. In the definition of $\tilde{S}(\omega)$, q^2 should be replaced by q_x^2 or optionally by $\boldsymbol{q}^2/3$. The simplest type of environment is known as the CL model, where the particle interacts with long wavelength modes. The associated form factor is

$$\tilde{S}_{\text{Caldeira–Leggett}}(q,\omega) = \tilde{S}_{\mathrm{ohmic}}(\omega) \times 3\frac{(2\pi)^3\delta^3(\boldsymbol{q})}{\boldsymbol{q}^2}. \tag{6}$$

Another case of interest is the interaction with a dirty metal (appendix B) for which

$$\tilde{S}_{\mathrm{Dirtymetal}}(q,\omega) \approx \tilde{S}_{\mathrm{ohmic}}(\omega) \times \frac{4\pi\ell^3}{\boldsymbol{q}^2} \qquad \text{for} \quad |\boldsymbol{q}| \lesssim \frac{1}{\ell}, \tag{7}$$

where the friction coefficient can be expressed in terms of the conductivity:

$$\eta = \frac{e^2}{\sigma} \times \frac{1}{4\pi\ell^3}. \tag{8}$$

In the latter context, it is customary to define a dimensionless friction coefficient as follows:

$$\alpha = \frac{1}{2\pi}\eta\ell^2 = \frac{e^2}{8\pi^2\sigma\ell} = \frac{3}{8(k_F\ell)^2}. \tag{9}$$

The motion of a classical Brownian particle of mass M under the influence of such a fluctuating environment is characterized by a damping rate

$$\gamma = \frac{\eta}{M} = \frac{2\pi\alpha}{M\ell^2}. \tag{10}$$

3. The fluctuations within a ring

In the present paper, we consider a particle in a ring of radius $L/(2\pi)$. We assume that $\tilde{S}(\boldsymbol{q},\omega) = \tilde{S}_{\rm ohmic}(\omega)\tilde{w}(\boldsymbol{q})$ is factorizable, as in the examples of equations (6) and (7). Since the motion is confined to one dimension, it is natural to expand the spatial correlations along the rings in Fourier series:

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\tilde{w}(\boldsymbol{q})\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot[\boldsymbol{R}(\theta_2)-\boldsymbol{R}(\theta_1)]} = \sum_{m=-\infty}^{\infty} w_m\,\mathrm{e}^{\mathrm{i}m(\theta_2-\theta_1)}. \tag{11}$$

Accordingly, using equation (2) with $x = (L/2\pi)\theta$ we get

$$\tilde{S}(q,\omega) = \tilde{S}_{\rm ohmic}(\omega) \times \sum_{m=-\infty}^{\infty} w_m 2\pi\delta(q-q_m), \tag{12}$$

where the discrete modes are

$$q_m = \frac{2\pi}{L}m, \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{13}$$

By convention, we want η to be the friction coefficient. Therefore, $\tilde{S}(\omega)$ as defined by equation (3) should equal $\tilde{S}_{\rm ohmic}(\omega)$ of equation (4). This implies the following sum rule:

$$\sum_{m=-\infty}^{\infty} w_m q_m^2 = 1. \tag{14}$$

In general, we have $\sim(L/\ell)$ fluctuating modes; each has the weight $w_n \sim \ell^3/L$. In appendix C, we show that for a CL bath we have only one fluctuating mode ($|m| = 1$) with

$$w_m = \frac{1}{2}\left(\frac{L}{2\pi}\right)^2 \tag{15}$$

while in the case of a dirty metal with short-range correlated fluctuations we have $\mathcal{M} = (L/(2\pi))/\ell \gg 1$ fluctuating modes with weights

$$w_m \approx \frac{\ell^2}{2\pi} \times \frac{1}{\mathcal{M}} \ln\left(\frac{\mathcal{M}}{|m|}\right) \qquad \text{for} \quad |m| < \mathcal{M}. \tag{16}$$

In both cases, we ignore the $m = 0$ mode for a reason which is explained in the following section. It is important to realize that the CL model can formally be regarded as a special limit of a dirty metal environment with $\ell \gg L$. In the latter case, the weight of the $|m| > 1$ modes is smaller by powers of L/ℓ (appendix C).

4. The dephasing factor

The dephasing factor P_φ is a number within $[0, 1]$ that characterizes the suppression of coherence. Its popular definition is based on a semiclassical picture. Using the Feynman–Vernon formalism, the propagator is expressed as a sum over pairs of classical trajectories. One observes that after time t, the interference contribution (from the off-diagonal terms in the double sum) is suppressed by a factor

$$P_\varphi(t) = |\langle U[x^A]\chi|U[x^B]\chi\rangle| = \mathrm{e}^{-S_N[x^A,x^B]}, \tag{17}$$

where χ is the preparation of the bath[2]. In order not to complicate the notations, the canonical average over χ states is implicit. The unitary operator $U[x]$ generates the evolution of the bath given that the particle goes along the trajectory $x(t)$. The action $S_N[x^A, x^B]$ is a double time integral. Using manipulation as in [11, 12], one obtains equation (1) with the symmetrized version of $\tilde{S}(q,\omega)$ and the symmetric classical version of $\tilde{P}(q,\omega)$. This semiclassical expression is definitely wrong for short-range scattering at low temperatures [12], because it does not reflect that *closed channels cannot be excited*. This problem with the semiclassical (stationary phase) approximation is well known in the theory of inelastic scattering. One way to overcome this limitation is to refine the definition of the dephasing factor using a semiclassically inspired 'scattering' point of view as in appendix D. However, it is clear that such a refinement makes the concept of dephasing too vague.

We propose here a natural definition for the dephasing factor that is related to the purity $\mathrm{trace}(\rho^2)$ of the reduced probability matrix. The notion of purity is very old, but in recent years it has become very popular due to the interest in quantum computation [23]. Given that the state of the system including the environment is Ψ_{pn}, where p and n label the basis states of the particle and the bath respectively, the purity is given by

$$P_\varphi(t) = \sqrt{\mathrm{trace}\left(\rho_{\mathrm{sys}}^2\right)} = \sqrt{\mathrm{trace}\left(\rho_{\mathrm{env}}^2\right)} = \left[\sum_{p'p''n'n''} \Psi_{p'n'}\Psi^*_{p''n'}\Psi_{p''n''}\Psi^*_{p'n''}\right]^{1/2}. \tag{18}$$

Assuming a factorized initial preparation as in the conventional Feynman–Vernon formalism, we propose the loss of purity ($P_\varphi < 1$) as a measure for decoherence. A standard reservation applies: initial transients during which the system gets 'dressed' by the environment should be ignored as these reflect renormalizations due to the interactions with the high frequency modes. Other choices of initial state might involve different transients, while the later slow approach to equilibrium should be independent of these transients. In any case the reasoning here is not much different from the usual ideology of the Fermi golden rule, which is used with similar restrictions to calculate transition rates between levels.

Writing the initial preparation as $\Psi^{(0)}_{pn} = \delta_{p,p_0}\delta_{n,n_0}$, and using leading order perturbation theory, we can relate P_φ to the probabilities $P_t(p,n|p_0,n_0) = |\Psi_{pn}|^2$ to have a transition from the state $|p_0,n_0\rangle$ to the state $|p,n\rangle$ after time t. The derivation is detailed in appendix E. One obtains the result

$$P_\varphi(t) = P_t(p_0,n_0|p_0,n_0) + P_t(p\neq p_0,n_0|p_0,n_0) + P_t(p_0,n\neq n_0|p_0,n_0), \tag{19}$$

in agreement with the semiclassically inspired point of view of appendix D. The notation $p \neq p_0$ or $n \neq n_0$ implies a summation $\sum_{p\neq p_0}$ or $\sum_{n\neq n_0}$, respectively. In the following section we shall discuss the actual calculation of $P_t(p,n|p_0,n_0)$, using the Fermi golden rule (FGR). Thus we deduce that within the FGR framework, the purity is simply the probability that either the system or the bath does not make a transition. The first term in equation (19) is just the survival probability of the preparation $P_{\mathrm{survival}} = P_t(p_0,n_0|p_0,n_0)$. The importance of the two other terms can be demonstrated using simple examples. For an environment that consists of static scatterers we have $P_{\mathrm{survival}} < 1$ but $P_\varphi = 1$, thanks to the second term. For a particle in a ring that interacts with a $q = 0$ environmental mode $P_{\mathrm{survival}} < 1$ but $P_\varphi = 1$, thanks to the third term. Using $\sum_{p,n} P_t(p,n|p_0,n_0) = 1$, we obtain the optional expression

$$p_\varphi = 1 - P_\varphi = \sum_{p\neq p_0}\sum_{n\neq n_0} P_t(p,n|p_0,n_0). \tag{20}$$

[2] The implicit assumption of an initial factorized state is of course problematic [24, 25]. In most cases, it implies an unpleasant transient that should be ignored. We further discuss the significance of the long time decay later in this section after equation (18) and in the summary.

In the problem that we consider in this paper, we can calculate P_φ using a $\mathrm{d}q\,\mathrm{d}\omega$ integral as in equation (1). In many examples the $\omega = 0$ transitions have zero measure, and therefore P_φ is practically the same as $P_t(p_0, n_0|p_0, n_0)$. Otherwise, one has to be careful in eliminating those transitions that do not contribute to the dephasing process. Anticipating the application of equation (1) for the calculation of the dephasing for a particle in a ring, the integration over q becomes a summation over q_m, and the $m = 0$ component should be excluded.

5. Dephasing at finite temperatures

The interaction between the particle ($\hat{x}$) and the environment can be written as in appendix A:

$$V = \int \mathrm{d}x\, \hat{\rho}(x)\hat{\mathcal{U}}(x), \tag{21}$$

where $\hat{\rho}(x) = \delta(x - \hat{x})$. In the Heisenberg (interaction) picture, a time index is added so we have $\hat{\mathcal{U}}(x, t)$ and $\hat{\rho}(x, t)$. Given a preparation of the bath and of the system, we can define $\tilde{S}(q, \omega)$ to characterize $\hat{\mathcal{U}}(x, t)$ and we can also define $\tilde{P}(q, \omega)$ to characterize $\hat{\rho}(x, t)$. The precise definition of the latter object is further discussed below. The survival probability of a factorized preparation is $P_\varphi(t) = 1-p_\varphi(t)$, where

$$\begin{aligned} p_\varphi(t) &= \int_0^t \int_0^t \langle V(t_2)V(t_1)\rangle \,\mathrm{d}t_2\,\mathrm{d}t_1 \\ &= \iint \mathrm{d}t_1\,\mathrm{d}t_2 \iint \mathrm{d}x_1\,\mathrm{d}x_2 \langle \rho(x_2, t_2)\mathcal{U}(x_2, t_2)\rho(x_1, t_1)\mathcal{U}(x_1, t_1)\rangle \\ &= \iint \frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}\omega}{2\pi}\tilde{S}(q, \omega) \iint \mathrm{d}t_1\,\mathrm{d}t_2 \iint \mathrm{d}x_1\,\mathrm{d}x_2 \langle \rho(x_2, t_2)\rho(x_1, t_1)\rangle\, \mathrm{e}^{\mathrm{i}q(x_2-x_1)-\mathrm{i}\omega(t_2-t_1)}. \end{aligned} \tag{22}$$

At finite temperatures, if recurrences due to the finite-size quantization effect can be ignored, one can obtain as an approximation $p_\varphi \approx \Gamma_\varphi t$, where Γ_φ is called the dephasing rate. In the following section, we discuss circumstances where such an approximation is not valid: the feasibility of this approximation requires neglect of the end-point contributions to the double time integration. By going to the variables $(t_1+t_2)/2$ and $\tau = t_2 - t_1$, one obtains the following expression for the dephasing rate:

$$\Gamma_\varphi = \iint \frac{\mathrm{d}q}{2\pi}\frac{\mathrm{d}\omega}{2\pi}\tilde{S}(q, \omega)\tilde{P}(-q, -\omega). \tag{23}$$

The implied definition of $\tilde{P}(q, \omega)$ is discussed below and further refined in the following section. Following standard argumentation one conjectures that the long time decay of $P_\varphi(t)$ is exponential, as in the analysis of Wigner's decay. The similarity of equation (23) to the semiclassical result (as discussed below equation (1)) is obvious. It is important to realize that in the present context, the *non*-symmetrized quantum version of the power spectrum has emerged. Furthermore, if we want to calculate P_φ, and not just the survival probability of the initial state, we have to be careful about the proper treatment of the diagonal terms as discussed in the previous section. Accordingly, we eliminate the diagonal term from the implied definition of the power spectrum:

$$\tilde{P}(q, \omega) = \int_{-\infty}^{+\infty} [\langle \mathrm{e}^{-\mathrm{i}qx(\tau)}\, \mathrm{e}^{\mathrm{i}qx(0)}\rangle - \langle \mathrm{e}^{\mathrm{i}qx}\rangle^2]\, \mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau. \tag{24}$$

We emphasize again that in a later section we are going to treat the time limits more carefully, where $\tilde{P}(q, \omega)$ will be replaced by $\tilde{P}(q, \omega; t)$ as in equation (1). For a ballistic particle with

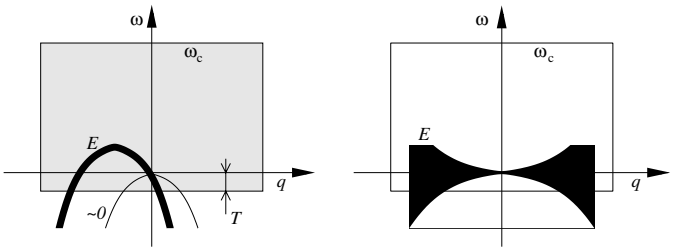

Figure 1. The (q, ω) plane. The power spectrum $\tilde{S}(q, \omega)$ is distributed pre-dominantly within the rectangular area $q \lesssim (1/\ell)$ that has a high frequency absorption cutoff ω_c and a lower emission cutoff. The emission cutoff T in this illustration reflects an assumption of having $T < \omega_c$, otherwise it would be equal to ω_c too. The power spectrum $\tilde{P}(q, \omega)$ which is associated with the ballistic motion (left panel) or with the diffusive motion (right panel) of the particle is illustrated by the dark region. In both cases the energy E of the particle implies a frequency cutoff, which is analogous to T. Close to equilibrium one should take $E \sim T$, but without much error we take for low temperatures ballistic motion $E \sim 0$, which is also illustrated in the left panel.

mass M and momentum $p = (2ME)^{1/2}$, we have

$$\tilde{P}(q, \omega) = 2\pi\delta(\omega - \omega(q)), \tag{25}$$

where $\omega(q) = [(p + q)^2 - p^2]/(2M)$. The power spectrum is illustrated in figure 1. The expectation value in equation (25) is taken for a particle with momentum p. For the ground state $p = 0$ and hence $\omega(q) = q^2/2M$. In particular, for a particle on a ring equation (12) implies $\omega(q_m) = q_m^2/2M$. The ballistic case should be contrasted (see figure 1) with the power spectrum of a diffusive particle:

$$\tilde{P}(q, \omega) = \frac{2Dq^2}{\omega^2 + (Dq^2)^2}, \tag{26}$$

where D is the diffusion coefficient. In the ballistic case the power spectrum is concentrated along $\omega = \omega(q)$, while in the diffusive case it spreads over the range $|\omega| < Dq^2$. In any case,

$$\int_{\omega \neq 0} \tilde{P}(q, \omega) \frac{\mathrm{d}\omega}{2\pi} = 1 \qquad \text{by definition, for any } q. \tag{27}$$

Assuming close-to-equilibrium conditions, the expectation value in equation (24) should reflect a thermal state with energy $E \sim T$. In practice one may set $E \sim 0$, though in general one should better be careful about it; looking at figure 1 one can deduce that taking $E \sim 0$, in the problem that we are going to consider, results in an underestimation of the dephasing rate by a $\sqrt{2}$ factor[3].

6. Dephasing at 'zero' temperature

The expression for Γ_φ manifestly gives a zero dephasing rate in the limit of zero temperature, because in this limit $\tilde{S}(q, \omega)$ and $\tilde{P}(-q, -\omega)$ have no overlap. However, this does not mean that P_φ does not decay. It still might have a sub-exponential decay. In order to understand this

[3] Assuming that $E \sim 0$, we are going to explain in section 7 that the non-negligible contribution to the integral comes from the range $|q| < q_T$ of effective modes; only the fluctuating modes in the rectangular region of figure 1 resonate with the particle and hence contribute. The power spectrum of the particle for $E \sim T$ is shifted 'upwards' in ω, and consequently the effective q range becomes larger by factor $\sqrt{2}$ compared with the $E \sim 0$ case.

point, we first discuss a simple artificial calculation of the double time integral

$$p_\varphi(t) = \int_0^t \int_0^t C(t_1 - t_2)\, \mathrm{d}t_1\, \mathrm{d}t_2, \tag{28}$$

where $C(\tau)$ is the symmetrized force–force correlation; its Fourier transform $\tilde{C}(\omega)$ is the symmetrized version of $\tilde{S}(\omega)$ of equation (4). Later, in this section, we come back to the actual calculation and discuss how equation (22) can be treated.

The approximation $p_\varphi(t) \approx \Gamma_\varphi t$ for the integral in equation (28) is based on the assumption that $C(\tau)$ has short-range correlations with a non-vanishing integral. At finite T, this assumption is indeed satisfied because $\tilde{S}(\omega = 0) = 2\eta T$ is finite. But at zero temperature, the integral over $C(\tau)$ is zero. In fact, at zero temperature, equation (4) implies that $\tilde{C}(\omega) = 2\eta|\omega|$ and hence $C(\tau)$ has power law tails $-(\eta/\pi)/\tau^2$. It is important to realize that the $T = 0$ behavior also prevails at finite temperatures provided $T < \omega_c$, and the time of interest should be smaller compared with $1/T$. Under such '$T = 0$', conditions $C(\tau)$ can be approximated by its $T = 0$ version. In order to see what comes out from equation (28), we observe that

$$\Gamma_\varphi(t) = \frac{\mathrm{d}}{\mathrm{d}t} p_\varphi(t) = \int_{-t}^{t} C(\tau)\, \mathrm{d}\tau. \tag{29}$$

For an Ohmic bath at 'zero temperature', the integral over the power law tails of $C(\tau)$ gives $\Gamma(t) \propto 1/t$; hence the spreading is logarithmic:

$$p_\varphi(t) = \frac{2}{\pi}\eta \ln(\omega_c t) + \text{const.} \tag{30}$$

It is instructive to make the same calculation in ω space. One realizes that

$$p_\varphi(t) = \int \frac{\mathrm{d}\omega}{2\pi} \tilde{C}(\omega) \left[\frac{\sin(\omega t/2)}{\omega/2}\right]^2, \tag{31}$$

which for $\tilde{C}(\omega) \propto |\omega|$ gives correctly the logarithmic spreading.

Without any approximation we can generalize the above treatment so as to handle equation (22), also taking into account the *non*-symmetrized nature of the spectral functions. Performing the $\mathrm{d}x_1\, \mathrm{d}x_2$ integration, we obtain $p_\varphi(t) = F(t)$ as in equation (1) where

$$\tilde{P}(q, \omega; t) = \frac{1}{t} \int_0^t \int_0^t \langle \mathrm{e}^{-\mathrm{i}qx(t_2)}\, \mathrm{e}^{\mathrm{i}qx(t_1)} \rangle \mathrm{e}^{\mathrm{i}\omega(t_2 - t_1)}\, \mathrm{d}t_1\, \mathrm{d}t_2. \tag{32}$$

In complete analogy with the way in which the environmental fluctuations have been treated, we express the correlator as a Fourier integral over $\tilde{P}(q, \omega)$, and then we are able to explicitly perform the $\mathrm{d}t_1\, \mathrm{d}t_2$ integration. The outcome of this procedure allows us to express the result as a convolution:

$$\tilde{P}(q, \omega; t) = \frac{1}{2\pi t} \left[\frac{\sin(\omega t/2)}{\omega/2}\right]^2 (*) \tilde{P}(q, \omega). \tag{33}$$

An optional compact way of writing the final result is

$$p_\varphi(t) = \int \mathrm{d}q \iint \frac{\mathrm{d}\omega}{2\pi} \frac{\mathrm{d}\omega'}{2\pi} \tilde{S}(q, \omega) \tilde{P}(-q, -\omega') \left[\frac{\sin((\omega - \omega')t/2)}{(\omega - \omega')/2}\right]^2. \tag{34}$$

We note that in practical calculations or for aesthetic reasons, it is possible to make the replacement

$$\left[\frac{\sin(\omega t/2)}{\omega/2}\right]^2 \longmapsto \left[\frac{(2/t)}{(1/t)^2 + \omega^2}\right] \times t \qquad \text{[optional]}. \tag{35}$$

The more convenient Lorentzian kernel has the same normalization, the same width $1/t$ and the same $2/\omega^2$ tails. It can be regarded as arising from using 'soft' rather than 'sharp' cutoff for the time integration.

7. Dephasing in the presence of a dirty metal '$T > 0$'

We turn to consider a particle of mass M in a ring of length L. We assume close-to-equilibrium conditions, so we take the energy of the particle above the ground state as $E \sim T$. We consider, in this section, temperatures T that are much larger compared with the level spacing $\Delta \sim (ML^2)^{-1}$. This does not mean that the system is not coherent[4]. We further assume that the time of interest is much longer compared with the relevant dynamical time scales, and in particular compared with $1/T$. With this assumption it is legitimate to use equation (23) to calculate the dephasing rate Γ_φ, and to treat the $d\omega$ integration as if the levels of the ring form a continuum. With the substitution of equations (12), (25), this leads to the following result:

$$\Gamma_\varphi = \sum_m w_m \tilde{S}_{\rm ohmic}(-\omega(q_m)). \tag{36}$$

Of course, one has to verify at the end of the calculation the self-consistency condition $\Gamma_\varphi \ll \min\{T, \omega_c\}$. This condition would be satisfied if the system–environment coupling is not too strong.

A graphic illustration of the (q, ω) integration domain is presented in figure 1. The T dependence of $\tilde{S}_{\rm ohmic}$ in equation (36) limits the sum to $\omega(q_m) < T$. Taking into account the weight factors, the effective number of fluctuating modes is

$$\mathcal{M}_{\rm eff} = \frac{L}{2\pi} q_{\rm eff} \approx \min\{\mathcal{M}, q_c L, q_T L, \}, \tag{37}$$

where $q_c = (2M\omega_c)^{1/2}$ and $q_T = (2MT)^{1/2}$. Note that if we had Fermi occupation it would be $q_T = T/v_{\rm F}$, while for diffusive motion it would be $q_T = (T/D)^{1/2}$. The dephasing rate is obtained by summing over all the contributing modes. Each effective mode contributes $2\eta T \times w_m$ to the sum. Accordingly,

$$\Gamma_\varphi = 2\eta T \times \sum_{0<|q_m|<q_{\rm eff}} w_m \sim 2\eta T \times \bar{w}\mathcal{M}_{\rm eff}, \tag{38}$$

where the average weight is $\bar{w} \sim \ell^2/\mathcal{M} \sim \ell^3/L$ for a short-range correlated dirty metal environment ($\ell \ll L$), while $\bar{w} \sim L^2$ in the opposite CL limit ($\ell \gg L$), as implied by equations (16) and (15) respectively. If all the modes are effective we get $\Gamma_\varphi = 2\eta T \times \ell^2$, while in the case of a CL environment we get the well-known result

$$\Gamma_\varphi = 2\eta T \times L^2 \qquad \text{[Caldeira–Leggett]}. \tag{39}$$

For a fluctuating environment with a correlation distance ℓ, equation (37) implies a crossover temperature:

$$T^* = \min\left\{\frac{1}{M\ell^2}, \omega_c\right\}. \tag{40}$$

For $T < T^*$ the dephasing rate depends on q_T and therefore develops nonlinear dependence on the temperature, as is illustrated in figure 2 and further discussed below. Using a field theoretical approach [18], it is argued that the renormalized value of the high frequency cutoff is

$$\omega_c|_{\rm effective} = \max\{\gamma, \Delta\}, \tag{41}$$

[4] As discussed in section 9, the coherence measure is $\Gamma_\varphi/\Delta_{\rm eff}$. In the regime of main interest $\Gamma_\varphi \ll T$. Furthermore, the relevant energy scale is not necessarily the level spacing; in the mesoscopic context the relevant energy scale (e.g. the 'Thouless energy') is typically much larger and proportional to $\hbar$ in contrast to the microscopic quantization scale which is proportional to $\hbar^{\rm dimensionality}$.

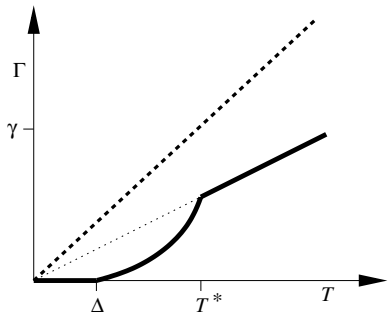

Figure 2. Illustration of the dependence of the dephasing rate Γ on the temperature T. The dephasing rate is well defined for $t > (1/T)$, and hence the self-consistency requirement is $\Gamma \ll T$. This condition is demonstrated by a comparison with the dashed line. The illustration reflects an assumption of having $\eta\ell^2 \ll 1$, and therefore the crossover temperature T^* is equal to the damping rate $\gamma = \eta/M$. The illustration further reflects an assumption of 'large ring' ($\eta L^2 \gg 1$) for which $\gamma \gg \Delta$, else the low temperature regime ($\Delta \ll T \ll T^*$) disappears. For extremely low temperatures, such that T is smaller compared with the spacing $\Delta = 1/(ML^2)$, the probability to excite the system is exponentially small and the familiar two-level modeling becomes applicable.

where γ is the classical damping rate of equation (10) and Δ is the level spacing. The reasoning is as follows: all the higher frequencies contribute only to mass renormalization and do not affect the dephasing process. In more details, the significant renormalization starts only below ω_c where the linear $|\omega|$ dispersion of the dissipation term dominates and leads to $\ln\omega$ terms in perturbation theory and to the need of either the renormalization group or an equivalent variational method [18].

We would like to remark that if we do not apply equation (41), the results that we derive below are affected quantitatively but not qualitatively. Substitution of equation (41) into equation (40) implies that $\eta\ell^2 < 1$ and $\eta\ell^2 > 1$ define distinct regimes of behavior. For a dirty metal environment, $\eta\ell^2 \ll 1$ is equivalent to $\alpha \ll 1$, i.e. $k_F\ell \gg 1$. For a CL environment $\eta\ell^2$ is formally infinite, or one may say that ℓ is effectively determined by the finite size L of the system.

We come back to the dephasing rate calculation. In the case of a fluctuating environment with a short correlation distance ℓ, we see that the high temperature ($T > T^*$) result is

$$\Gamma_\varphi \approx \begin{cases} (2\eta\ell^2)T & \text{if} \quad \eta\ell^2 \gg 1 \\ (2\eta\ell^2)^{3/2}T & \text{if} \quad \eta\ell^2 \ll 1 \end{cases} \qquad [\text{for } T > T^*], \tag{42}$$

where in the $\eta\ell^2 \ll 1$ expression we have identified the effective (renormalized) cutoff as $\omega_c = \gamma$. The strong coupling result ($\eta\ell^2 \gg 1$) cannot be trusted because the self-consistency requirement ($\Gamma_\varphi \ll T$) is not satisfied. This is not in contradiction with the observation that the CL result (39) is formally a special case of the strong coupling result with $\ell \mapsto L$. In the latter case, the self-consistency relation becomes $\eta L^2 \ll 1$ irrespective of ℓ.

In the low temperature regime, we have (be definition) $q^* = q_T$. Consequently the T dependence becomes nonlinear, and we get $\Gamma_\varphi \approx \eta\ell^3 M^{1/2} T^{3/2}$. The similarity of the latter to the familiar result for a diffusive electron is misleading. In both cases $q_T \propto T^{1/2}$ but for different reasons, and with different prefactors. For sake of completeness, we write the precise expression which is obtained for a dirty metal environment using equation (38) with equations (16) and (8):

$$\Gamma_\varphi = \frac{e^2}{4\pi^2\sigma} T q_T \ln\left(\frac{1}{q_T\ell}\right) \qquad [\text{for } \Delta < T < T^*]. \tag{43}$$

Since $\Gamma_\varphi \sim T^{3/2}$ at sufficiently low T, the condition $\Gamma_\varphi \ll T$ is valid even for strong coupling $\eta\ell^2 > 1$. We also note that inclusion of a $q = 0$ mode in (38) would have led to $\Gamma_\varphi \sim w_0 T$. Hence, our precise formulation in equation (45) is essential.

The crossover from the high temperature result to the low temperature result is illustrated in figure 2. The illustration assumes $\eta\ell^2 \ll 1$, which implies that the self-consistency requirement ($\Gamma_\varphi \ll T$) is globally satisfied. It also should be realized that the low temperature regime $\Delta \ll T \ll T^*$ exists only for a 'large ring' ($\eta L^2 \gg 1$), for which $T^* = \gamma \gg \Delta$. Finally, for the $L < \ell$ case, the weight $w_{\pm 1}$ dominates, leading to $\mathcal{M}_{\rm eff} = 1$ and $\Gamma_\varphi = 2\eta L^2 T$ as in equation (39), which for $\omega_c < \Delta$ is consistent with $\Gamma_\varphi < \Delta$ for $\eta L^2 < 1$.

8. Dephasing in the presence of a dirty metal '$T = 0$'

We would like to discuss the 'zero temperature' regime. One should be very careful in specifying the conditions of physical interest, else the problem may become trivial or of no experimental relevance. In what follows, we assume that the dimensionless coupling between the system and the environment ($\eta\ell^2$ for a dirty metal or ηL^2 for a CL environment) is much smaller than unity. This means that the competing energy scales are the level spacing $\Delta \sim 1/(ML^2)$ and the temperature. So the simplest definition of zero temperature is $T \ll \Delta$ for which the system is in the ground state with an exponentially small probability to be found in an excited state. In this regime, the mass renormalization effect can be calculated using the second-order perturbation theory, as in the Polaron problem, or possibly using field theoretical methods. Furthermore in this regime we can treat the dephasing problem using a 'two level approximation', which is a very well-studied model [25].

The notion of 'zero temperature' is also applicable if $T \gg \Delta$ provided the time of interest is short ($t \ll 1/T$). In this regime, the power spectrum $\tilde{S}(q, \omega)$ is the same as for $T = 0$ within the frequency interval $T \ll \omega \ll \omega_c$. Consequently, the Ohmic temporal correlations are $C(\tau) \approx -(\eta/\pi)/\tau^2$ within the time interval $(1/\omega_c) \ll t \ll (1/T)$. As explained in a previous section, such correlations may imply a logarithmic growth of $p_\varphi(t)$. In view of the claim that the renormalized value of ω_c is the damping rate γ, it follows that logarithmic spreading may arise only if $T \ll \gamma$, which is the low temperature regime.

As discussed in a previous section, equation (34) gives a non-zero result for $p_\varphi(t)$ even at zero temperature. For the CL model, we have only $q \sim 0$ fluctuating modes and we get the expected $\log(t)$ spreading:

$$p_\varphi(t) = \frac{\eta}{\pi}\left(\frac{L}{2\pi}\right)^2 \ln(\omega_c t) \qquad \text{[Caldeira–Leggett]}, \tag{44}$$

where ω_c is the high frequency cutoff of the temporal fluctuations. More generally, for a particle in a ring the result can be written as a sum over all the q Fourier components:

$$p_\varphi(t) = \frac{\eta}{\pi}\sum_m w_m \ln\left(\frac{\omega_c}{(1/t) + \omega(q_m)}\right). \tag{45}$$

Strictly speaking, for a finite system, the assumption $T \gg \Delta$ always breaks down in the zero temperature limit. Still it is meaningful to formulate a condition for *not* having dephasing at zero temperature irrespective of the finite size effect:

$$\lim_{L\to\infty} p_\varphi(t = \infty) \ll 1. \tag{46}$$

Thus, the question is simply whether in the continuum limit the q summation in equation (45) converges in its lower limit. For a fluctuating environment with a finite (short) correlation

distance ℓ,

$$p_\varphi \sim \int_0^{1/\ell} \mathrm{d}q\, \eta\ell^3 \ln\left[\frac{M\omega_c}{q^2}\right] \sim \eta\ell^2 \ln\left(\frac{\omega_c\ell^2}{M}\right). \tag{47}$$

We conclude that for a dirty metal environment with $\ell \ll L$, coherence is maintained if $\eta\ell^2 \ll 1$, i.e. if $k_F\ell \gg 1$.

It is important to note the following: in the strict Caldeira–Leggett limit ($\ell = \infty$), the size of the ring L can be arbitrarily large; hence the Heisenberg time $1/\Delta$ becomes huge, and equation (45) leads to equation (44), which becomes an exact result. So if we consider a *dirty metal environment* with long wavelength fluctuations ($\ell \gg L$), it looks as if we are back in the 'CL regime' leading to equation (44). But this is not quite correct unless we give away the weak coupling assumption $\eta\ell^2 \ll 1$. As long as we keep $\alpha \ll 1$ (fixed), the CL result does not apply. This is because once $\ell \to L$ and $\eta L^2 < 1$ the quantization of the energy spectrum becomes important, and the renormalized cutoff frequency (41) becomes $\omega_c \sim \Delta$ instead of $\omega_c \sim \gamma$. Accordingly, in the latter case, the time during which the log spreading prevails diminishes.

9. Mass renormalization

It can be shown [10] that a particle that interacts with a fluctuating 'dirty' environment acquires an additional inertial (polaronic) mass. However in recent works [16–18], the mass-renormalization concept appears in a new context. The free energy $\mathsf{F}(T, \Phi)$ of a particle in a ring is calculated, where T is the equilibrium temperature and Φ is the Aharonov Bohm flux through the ring. Then the coherence is characterized by the 'curvature', which is a measure for the sensitivity to Φ. If the interaction with the environment is negligible, the result can be written as

$$\left.\frac{\partial^2\mathsf{F}}{\partial\Phi^2}\right|_{\Phi=0} = \frac{e^2}{M^*L^2} f(M^*L^2T) \tag{48}$$

with the bare mass $M^* = M$. The dependence of the curvature on T merely reflects the Boltzmann distribution of the energy. In the presence of coupling to the environment, it turns out that $M^* > M$. At $T = 0$, for fixed $\eta\ell^2 \ll 1$, Monte Carlo data show [22] that the ratio M^*/M is independent of the radius beyond a critical L_c. As the radius becomes smaller than L_c, the ratio M^*/M rapidly approaches unity. In the regime of 'large L' the mass-renormalization effect diminishes with the temperature and depends on the scaled variable LT, while for 'small L' the ratio M^*/M grows with the temperature and depends on the scaled variable L^4T.

The natural question is whether we can shed some light on the physics behind this observed temperature dependence of the mass renormalization factor. In particular we would like to explain why in in one regime M^*/M is a function of LT, while in another regime it is a function of L^4T. Making the conjecture that the temperature dependence of M^*/M is determined by dephasing, it is natural to suggest the following measure of coherence:

$$x(T, L) = p_\varphi\left(t = \frac{1}{\Delta_{\mathrm{eff}}}\right) = \frac{\Gamma_\varphi}{\Delta_{\mathrm{eff}}}. \tag{49}$$

Namely, it is the dephasing factor at the time $t = 1/\Delta_{\mathrm{eff}}$, where Δ_{eff} is the 'relevant' energy scale. Equivalently, the condition $x \ll 1$ means that the energy levels near Δ_{eff} remain sharp. The inequality $x < 1/2$ can serve as a practical definition for having coherence. Either it can be translated as a condition on the temperature or optionally it can be used in order to define a coherence length that depends on the temperature. The conjecture is that $y = M^*/M$ is a

function of x. Let us calculate x using equation (38). We assume $\eta\ell^2 \ll 1$, but ηL^2 can be either larger or smaller compared to unity. This is equivalent to saying that the damping rate γ can be either larger or smaller compared with the level spacing Δ. Using equation (40) with equation (41), this further implies that $\mathcal{M}_{\rm eff}$ is either larger or of order unity respectively. The transitions that are associated with the 'relevant' energy levels are characterized by $q \sim q_{\rm eff}$ and accordingly $\Delta_{\rm eff} \sim \mathcal{M}_{\rm eff} \times (ML^2)^{-1}$. Using equation (38) we deduce that the result does not depend on $\mathcal{M}_{\rm eff}$ but only on $\bar{w}$, leading to

$$x(T, L) = \eta\bar{w}ML^2T = \begin{cases} \eta M\ell^3 LT & \text{for} \quad L \gg \ell \\ \eta ML^4T & \text{for} \quad L \ll \ell. \end{cases} \tag{50}$$

We recall that our Γ_φ is valid at least for weak coupling $\eta\ell^2 \ll 1$. The scaling of Γ_φ with L is consistent with Monte Carlo exponents for the coherence length $L \sim T^{-\mu}$ with either $\mu = 1$ or $\mu = 1/4$. The Monte Carlo data have not determined so far whether the transition between the two regimes is at $L \approx \ell$ or whether it is coupling dependent.

10. Summary and discussion

In this paper, we have defined and calculated the dephasing factor $P_\varphi(t)$ for a particle in a ring due to the fluctuations of a dirty metal environment. At finite temperature, we have calculated the dephasing rate Γ_φ. Our interest was mainly in the mesoscopic regime $\Delta \ll \Gamma_\varphi \ll \gamma$, where interference is important (because $\Gamma_\varphi \ll \gamma$). Unlike the microscopic regime ($\Gamma_\varphi \ll \Delta$), which is customary in atomic physics studies, here the question of dephasing at low temperature is tricky both conceptually and technically.

The decoherence is induced because the system gets entangled with the environmental modes. It should be clear that under generic circumstances, the coupling always induces 'transitions' that lead to system-bath entanglement. Accordingly, we have $P_\varphi(t) < 1$ even if '$T = 0$'. This by itself does not mean 'having dephasing': entanglement is also associated with the adiabatic renormalization due to the interaction with the high frequency modes. In order to 'have dephasing', the loss of purity should not be just a *transient*; rather it should be a progressive process.

Still even with this careful point of view, the reader may doubt whether the notion of the 'dephasing factor' is really helpful in studying dephasing. After all what do we 'really' mean by dephasing? Possibly the 'correct' procedure is to study an equilibrium correlation function $C(t)$, and to ask whether it is damped in the $t \to \infty$ limit. In the absence of coupling to the environment, the Fourier transform $\tilde{C}(\omega)$ is a sum over delta functions $\delta(\omega - \Omega_r)$. Due to the coupling, the deltas are broadened into resonances with Γ_r. This is true at any temperature, also at '$T = 0$'. The controversy about dephasing at '$T = 0$' is related to the limit $L \to \infty$. Do the resonances overlap in this limit? Do singular features of the uncoupled system survive? For sub-Ohmic bath [25] the ratio Γ_φ/Δ, where Δ is the mean level spacing, diverges as $L \to \infty$. But the Ohmic case is 'marginal' and within the framework of the FGR it remains a constant α. So if this α is smaller compared with unity, we naively expect no dephasing at '$T = 0$'.

The naive expectation of having no dephasing at '$T = 0$' is not without loopholes. One obvious loophole is the mass renormalization issue. If hypothetically the renormalized mass and hence the density of states diverge as $T \to 0$, it might imply dephasing at zero temperature. The recent studies of equilibrium properties of the ring problem are aimed at studying this question carefully, in a controlled way. For a particle that interacts with a dirty metal environment we believe, on the basis of [18, 22], that the renormalized mass at zero

temperature is finite. So in the case of a dirty metal environment, there is no indication for 'dephasing at $T = 0$'.

Still one would like to know what happens at low, but finite temperature. As we said previously, no doubt that the study of equilibrium properties is conceptually the best procedure. Still, we also want to physically understand the results. Here we come back to the 'dephasing factor' notion. In spite of the problems which are associated with this concept, we believe that it is powerful enough to shed light on the physics of dephasing. Our aim in this paper was to maximally exploit this notion, within the Fermi golden rule picture, in order to demonstrate that it captures the correct physics of all the established results regarding dephasing. In particular, it has provided an explanation for the T dependence of the mass renormalization effect, and under what conditions the spatial aspect of the fluctuations is capable of suppressing the '$T = 0$' power law decay of coherence.

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Appendix A. Modeling of a fluctuating environment

It is customary to write the system-environment Hamiltonian as

$$\mathcal{H}_{\text{total}} = \mathcal{H}_{\text{sys}}(\hat{x},\, \hat{p}) + V_{\text{int}}(\hat{x},\, \hat{Q}_\alpha) + \mathcal{H}_{\text{env}}(\hat{Q}_\alpha,\, \hat{P}_\alpha), \tag{A.1}$$

where $(\hat{x},\, \hat{p})$ are the canonical coordinates of the particle and $(\hat{Q}_\alpha,\, \hat{P}_\alpha)$ are the environmental degrees of freedom. In the case of an interaction of a particle with a dirty metal environment (in three dimensions),

$$V_{\text{int}} = \int \mathrm{d}^3x\, \hat{\rho}(\boldsymbol{x}) \int \mathrm{d}^3x'\, \frac{e^2\hat{\mathsf{n}}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} \equiv \int \mathrm{d}^3x'\, \hat{\rho}(\boldsymbol{x})\hat{\mathcal{U}}(\boldsymbol{x}), \tag{A.2}$$

where the electronic density $\mathsf{n}(\boldsymbol{x})$ can be expressed as a function of their coordinates, while $\hat{\rho}(\boldsymbol{x}) = \delta(\boldsymbol{x} - \hat{\boldsymbol{x}})$ is a particle-related field operator.

In order to allow a Feynman–Vernon treatment, it is more convenient to regard $\mathcal{U}(\boldsymbol{x})$ as arising from the interaction with a bath of harmonic oscillators [10]. Each harmonic oscillator is a scatterer which is characterized by its location x_α and its natural frequency ω_α. The interaction of the particle with the α scatterer is $\hat{Q}_\alpha u(\hat{x} - x_\alpha)$, so we write (in one dimension)

$$V_{\text{int}} = \sum_\alpha c_\alpha \hat{Q}_\alpha u(\hat{x} - x_\alpha) = \int \mathrm{d}x\, \hat{\rho}(x)\hat{\mathcal{U}}(x), \tag{A.3}$$

where c_α are coupling constants, and $\hat{\rho}(x) = \delta(\hat{x} - x)$. In the Heisenberg (interaction) picture, a time index is added so we have $\hat{\rho}(x,t)$ and $\hat{\mathcal{U}}(x,t)$. Accordingly, the fluctuating filed is

$$\hat{\mathcal{U}}(x,t) = \sum_\alpha c_\alpha \hat{Q}_\alpha(t) u(x - x_\alpha). \tag{A.4}$$

As explained in [10] it is possible to postulate the interaction $u(r)$, and the distribution of the parameters $(x_\alpha, \omega_\alpha, c_\alpha)$, such as to obtain a fluctuating field with a physically desired $\tilde{S}(q,\omega)$. This type of modeling is equivalent to the field-theoretical assumption of having Gaussian fluctuations, and accordingly a linear response treatment of the environment becomes exact.

Appendix B. The fluctuations of a dirty metal

For a metal we can use FD relation in order to relate the spati-temporal power spectrum $\tilde{S}(q,\omega)$ to the conductivity:

$$\tilde{S}(\boldsymbol{q},\omega) = \frac{4\pi e^2}{\boldsymbol{q}^2}\mathrm{Im}\left[\frac{-1}{\varepsilon(\boldsymbol{q},\omega)}\right]\frac{2\hbar}{1-\mathrm{e}^{-\hbar\omega/T}}, \tag{B.1}$$

where

$$\varepsilon(\boldsymbol{q},\omega) = 1 + \frac{4\pi\sigma}{-\mathrm{i}\omega + D\boldsymbol{q}^2} \tag{B.2}$$

and

$$\mathrm{Im}\left[\frac{1}{\varepsilon(\boldsymbol{q},\omega)}\right] = -\frac{4\pi\sigma\omega}{(D\boldsymbol{q}^2+4\pi\sigma)^2+\omega^2}. \tag{B.3}$$

Thus, we get

$$\tilde{S}(\boldsymbol{q},\omega) \approx \frac{e^2}{\sigma}\frac{1}{\boldsymbol{q}^2}\frac{2\hbar\omega}{1-\mathrm{e}^{-\hbar\omega/T}} \qquad \text{for} \quad |\omega| \lesssim \omega_c, \quad |\boldsymbol{q}| \lesssim \frac{1}{\ell} \tag{B.4}$$

The Ohmic behavior is cut off by the Drude collision frequency ω_c, and the elastic mean free path is $\ell = v_\mathrm{F}/\omega_c$, where v_F is the Fermi velocity. The expression for η in equation (8) is obtained from the Kubo formula (5).

Appendix C. Calculation of Fourier components

We are interested only in fluctuations within the ring. Therefore, we have to calculate the Fourier components of the correlator $\langle\hat{\mathcal{U}}(x(\theta_2),t_2)\hat{\mathcal{U}}(x(\theta_1),t_1)\rangle$. For the CL model, we use the integral

$$\int \mathrm{d}^3\boldsymbol{q}\frac{3\delta^3(\boldsymbol{q})}{\boldsymbol{q}^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}} = \mathrm{const} - \frac{1}{2}\boldsymbol{R}^2 = \mathrm{const} + \left(\frac{L}{2\pi}\right)^2\cos(\theta_2-\theta_1), \tag{C.1}$$

where $\boldsymbol{R} = \boldsymbol{R}(\theta_2)-\boldsymbol{R}(\theta_1)$ so that $|\boldsymbol{R}| = |2\sin((\theta_2-\theta_1)/2)|[L/2\pi]$. For a dirty metal with fluctuations within $q \lesssim 1/\ell$, we have

$$\int\frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\frac{4\pi\ell^3}{\boldsymbol{q}^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}} = \frac{\ell^3}{\sqrt{\boldsymbol{R}^2+\ell^2}} = \ell^2\left[a_0 + \sum_{m=1}^{\infty} a_m\cos(m(\theta_2-\theta_1))\right]. \tag{C.2}$$

Our $a_{m\neq 0}$ are half the 'convention' in [16]. From the Fourier transform relation it follows that $\sum_{m=0}^{\infty} a_m = 1$, and we also have the sum rule $\sum_{m=1}^{\infty} a_m m^2 = \mathcal{M}^2$ where $\mathcal{M} = (L/(2\pi))/\ell$. Disregarding the $m = 0$ Fourier component, the following approximation can be obtained [16, 17] for $\mathcal{M} \gg 1$:

$$a_m \approx \frac{1}{\pi\mathcal{M}}\ln\left(\frac{\mathcal{M}}{m}\right) \qquad \text{for} \quad 0 < m < \mathcal{M}. \tag{C.3}$$

From equation (11), it follows that $w_m = \ell^2 a_m/2$. We conclude that the particle in the ring experiences a *white* fluctuating field that is characterized by a correlation distance ℓ. The fluctuating field can be reinterpreted as arising from a short-range interaction $u(r)$ with a uniformly distributed set of scatterers as in equation (A.4).

In the other extreme case of CL-like environment ($\ell \gg L$), the fluctuations of the higher ($m > 1$) modes are negligible compared with the fluctuations of the lowest ($m = 1$) mode. Accordingly, we say that the number of effective modes is $\mathcal{M} = 1$. Using the ad hoc notation $\bar{\mathcal{M}} = (L/(2\pi))/\ell \ll 1$, the sum rule which is based on equation (C.2) implies that $a_1 = \bar{\mathcal{M}}^2$ while $a_{m>1}$ have higher powers of $\bar{\mathcal{M}}$.

Appendix D. The dephasing factor—semiclassical perspective

For short-range scattering, if x^A and x^B of equation (17) are well separated, and hence interact with *different* sets of oscillators, we can argue within the semiclassical framework that P_φ is the probability to induce an excitation in the bath (i.e. 'to leave a trace in the environment'). The argument is elaborated in Appendix C of [11]. This argument *fails* if the interfering states are not well separated in space, but rather interact with the *same* oscillators. For this reason, the definition of the dephasing factor has to be refined. One possibility is to adopt a 'scattering' point of view, hence treating correctly closed channels. Using sloppy notations the idea is to define the dephasing factor in analogy with equation (17) as $P_\varphi = |\langle U[\psi^A]\chi|U[\psi^B]\chi\rangle|$, where ψ^A and ψ^B are *ingoing states* of the system. This way of writing is suggestive rather than exact. Referring to a superposition preparation of the ring, where ψ^A and ψ^B are momentum eigenstates, it is clear that P_φ is not necessarily the same as the probability to induce an excitation in the bath. This is because $U[\psi^A]$ and $U[\psi^B]$ involve the excitation of the *same* oscillators, rather than *different* sets of oscillators. If the factorized preparation is $|p_0 n_0\rangle$, then we write the evolved state in the interaction picture after time t as $|(p_0 n_0)_t\rangle$. If we have initially a superposition $|p_1\rangle + |p_2\rangle$, the evolved state would be

$$|\Psi\rangle = |p_1\rangle \otimes |\chi^{(1)}\rangle + |p_2\rangle \otimes |\chi^{(2)}\rangle + \text{inelastic part}, \tag{D.1}$$

where the so-called relative states of the bath are

$$\chi_n^{(1)} = \langle p_1 n|(p_1 n_0)_t\rangle \tag{D.2}$$

$$\chi_n^{(2)} = \langle p_2 n|(p_2 n_0)_t\rangle. \tag{D.3}$$

The dephasing factor is

$$P_\varphi = |\langle\chi^{(1)}|\chi^{(2)}\rangle| = P_t(p_0, n_0|p_0, n_0) + \sum_{n(\neq n_0)} \chi_n^{(1)*}\chi_n^{(2)}, \tag{D.4}$$

where we assume $p_1 \sim p_2 \sim p_0$. It is not difficult to realize that the same approximation implies that the second term equals $P_t(p_0, n \neq n_0|p_0, n_0)$ in agreement with equation (19). With some further argumentation, we can justify the second term in equation (19) as well. We note that this derivation parallels the semiclassical treatment in appendix D of [11], where $\tilde{P}(q,\omega)$ is defined as the difference $\tilde{P}_\parallel(q,\omega) - \tilde{P}_\perp(q,\omega)$.

Appendix E. The purity-based definition of the dephasing factor

In this appendix, we explain the derivation of equation (19) from equation (18). The zero-order term in equation (18) is the $p' = p'' = p_0, n' = n'' = n_0$ term. It is equal to P_0^2, where $P_0 = P_t(p_0, n_0|p_0, n_0)$. There are four sets of first-order terms. The sum of $p' = p'' = p_0, n' = n_0, n'' \neq n_0$ terms is $P \times p_{\text{sys}}$ where $p_{\text{sys}} = P_t(p_0, n \neq n_0|p_0, n_0)$. Here, $n \neq n_0$ implies a summation $\sum_{n\neq n_0}$. The sum of the $p' = p'' = p_0, n' \neq n_0, n'' = n_0$ terms is the same. There are two other sets, with either $p' \neq p_0$ or $p'' \neq p_0$, that give each $P \times p_{\text{env}}$, where $p_{\text{env}} = P_t(p \neq p_0, n_0|p_0, n_0)$. Summing over all the leading order contributions, we get

$$P_\varphi = \left[P_0^2 + 2P_0 \times p_{\text{sys}} + 2P_0 \times p_{\text{env}} + \mathcal{O}(p^2)\right]^{1/2} \tag{E.1}$$

leading to $P_\varphi \approx P_0 + p_{\text{sys}} + p_{\text{env}}$ which is equation (19).

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